CLASSIFYING FINITE LOCALIZATIONS OF QUASI-COHERENT SHEAVES

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ABSTRACT. Given a quasi-compact, quasi-separated scheme X, a bijection between the tensor localizing subcategories of finite type in $\mathrm{Qcoh}(X)$ and the set of all subsets $Y \subseteq X$ of the form $Y = \bigcup_{i\in\Omega} Y_i$, with $X \setminus Y_i$ quasi-compact and open for all $i \in \Omega$, is established. As an application, there is constructed an isomorphism of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{Qcoh}(X)), \mathcal{O}_{\mathrm{Qcoh}(X)}),$$

where $(\mathsf{Spec}(\mathrm{Qcoh}(X)), \mathcal{O}_{\mathrm{Qcoh}(X)})$ is a ringed space associated to the lattice of tensor localizing subcategories of finite type. Also, a bijective correspondence between the tensor thick subcategories of perfect complexes $\mathcal{D}_{\mathrm{per}}(X)$ and the tensor localizing subcategories of finite type in $\mathrm{Qcoh}(X)$ is established.

CONTENTS

1. INTRODUCTION

In his celebrated work on abelian categories P. Gabriel [6] proved that for any noetherian scheme X the assignments

$$\text{(1.1)} \quad \mathrm{coh}\,X \supseteq \mathcal{D} \mapsto \bigcup_{x\in\mathcal{D}} \mathsf{supp}_X(x) \quad \text{and} \quad X \supseteq U \mapsto \{x \in \mathrm{coh}\,X \mid \mathsf{supp}_X(x) \subseteq U\}$$

induce bijections between

(1) the set of all tensor Serre subcategories of $\mathrm{coh}\,X$, and
(2) the set of all subsets $U \subseteq X$ of the form $U = \bigcup_{i\in\Omega} Y_i$ where, for all $i \in \Omega$, Y_i has quasi-compact open complement $X \setminus Y_i$.

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As a consequence of this result, X can be reconstructed from its abelian category, $\operatorname{coh} X$, of coherent sheaves (see Buan-Krause-Solberg [4, Sec. 8]). Garkusha and Prest [8, 9, 10] have proved similar classification and reconstruction results for affine and projective schemes.

Given a quasi-compact, quasi-separated scheme X, let $\mathcal{D}_{\text{per}}(X)$ denote the derived category of perfect complexes. It comes equipped with a tensor product $\otimes := \otimes_X^L$. A thick triangulated subcategory $\mathcal{T}$ of $\mathcal{D}_{\text{per}}(X)$ is said to be a tensor subcategory if for every $E \in \mathcal{D}_{\text{per}}(X)$ and every object $A \in \mathcal{T}$, the tensor product $E \otimes A$ also is in $\mathcal{T}$. Thomason [26] establishes a classification similar to (1.1) for tensor thick subcategories of $\mathcal{D}_{\text{per}}(X)$ in terms of the topology of X. Hopkins and Neeman (see [15, 22]) did the case where X is affine and noetherian.

Based on Thomason's classification theorem, Balmer [1] reconstructs the noetherian scheme X from the tensor thick triangulated subcategories of $\mathcal{D}_{\text{per}}(X)$. This result has been generalized to quasi-compact, quasi-separated schemes by Buan-Krause-Solberg [4].

The main result of this paper is a generalization of the classification result by Garkusha and Prest [8, 9, 10] to schemes. Let X be a quasi-compact, quasi-separated scheme. Denote by $\operatorname{Qcoh}(X)$ the category of quasi-coherent sheaves. We say that a localizing subcategory $\mathcal{S}$ of $\operatorname{Qcoh}(X)$ is of finite type if the canonical functor from the quotient category $\operatorname{Qcoh}(X)/\mathcal{S} \to \operatorname{Qcoh}(X)$ preserves directed sums.

Theorem (Classification)**.** *Let X be a quasi-compact, quasi-separated scheme. Then the maps*

$$V \mapsto \mathcal{S} = \{\mathcal{F} \in \operatorname{Qcoh}(X) \mid \mathsf{supp}_X(\mathcal{F}) \subseteq V\}$$

and

$$\mathcal{S} \mapsto V = \bigcup_{\mathcal{F} \in \mathcal{S}} \mathsf{supp}_X(\mathcal{F})$$

induce bijections between

(1) *the set of all subsets of the form $V = \bigcup_{i \in \Omega} V_i$ with quasi-compact open complement $X \setminus V_i$ for all $i \in \Omega$,*
(2) *the set of all tensor localizing subcategories of finite type in* $\operatorname{Qcoh}(X)$.

As an application of the Classification Theorem, we show that there is a 1-1 correspondence between the tensor finite localizations in $\operatorname{Qcoh}(X)$ and the tensor thick subcategories in $\mathcal{D}_{\text{per}}(X)$ (cf. [16, 8, 10]).

Theorem. *Let X be a quasi-compact and quasi-separated scheme. The assignments*

$$\mathcal{T} \mapsto \mathcal{S} = \{\mathcal{F} \in \operatorname{Qcoh}(X) \mid \mathsf{supp}_X(\mathcal{F}) \subseteq \bigcup_{n \in \mathbb{Z}, E \in \mathcal{T}} \mathsf{supp}_X(H_n(E))\}$$

and

$$\mathcal{S} \mapsto \{E \in \mathcal{D}_{\text{per}}(X) \mid H_n(E) \in \mathcal{S} \text{ for all } n \in \mathbb{Z}\}$$

induce a bijection between

(1) *the set of all tensor thick subcategories of $\mathcal{D}_{\text{per}}(X)$,*
(2) *the set of all tensor localizing subcategories of finite type in* $\operatorname{Qcoh}(X)$.

Another application of the Classification Theorem is the Recostruction Theorem. A common approach in non-commutative geometry is to study abelian or triangulated categories and to think of them as the replacement of an underlying scheme. This idea goes back to work of Grothendieck and Manin. The approach is justified by the fact that a noetherian scheme can be reconstructed from the abelian category of coherent sheaves (Gabriel [6]) or from the category of perfect complexes (Balmer [1]). Rosenberg [24] proved that a quasi-compact scheme X is reconstructed from its category of quasi-coherent sheaves.

In this paper we reconstruct a quasi-compact, quasi-separated scheme X from $\mathrm{Qcoh}(X)$. Our approach, similar to that used in [8, 9, 10], is entirely different from Rosenberg's [24] and less abstract.

Following Buan-Krause-Solberg [4] we consider the lattice $L_{\mathrm{f.loc},\otimes}(X)$ of tensor localizing subcategories of finite type in $\mathrm{Qcoh}(X)$ as well as its prime ideal spectrum $\mathsf{Spec}(\mathrm{Qcoh}(X))$. The space comes naturally equipped with a sheaf of commutative rings $\mathcal{O}_{\mathrm{Qcoh}(X)}$. The following result says that the scheme $(X,\mathcal{O}_X)$ is isomorphic to $(\mathsf{Spec}(\mathrm{Qcoh}(X)),\mathcal{O}_{\mathrm{Qcoh}(X)})$.

Theorem (Reconstruction). *Let X be a quasi-compact and quasi-separated scheme. Then there is a natural isomorphism of ringed spaces*

$$f:(X,\mathcal{O}_X)\xrightarrow{\sim}(\mathsf{Spec}(\mathrm{Qcoh}(X)),\mathcal{O}_{\mathrm{Qcoh}(X)}).$$

Other results presented here worth mentioning are the theorem classifying finite localizations in a locally finitely presented Grothendieck category $\mathcal{C}$ (Theorem 3.5) in terms of some topology on the injective spectrum $\mathsf{Sp}\,\mathcal{C}$, generalizing a result of Herzog [13] and Krause [19] for locally coherent Grothendieck categories, and the Classification and Reconstruction Theorems for coherent schemes.

2. Localization in Grothendieck categories

The category $\mathrm{Qcoh}(X)$ of quasi-coherent sheaves over a scheme X is a Grothendieck category (see [5]), so hence we can apply the general localization theory for Grothendieck categories which is of great utility in our analysis. For the convenience of the reader we shall recall some basic facts of this theory.

We say that a subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ is a *Serre subcategory* if for any short exact sequence

$$0\to X\to Y\to Z\to 0$$

in $\mathcal{C}$ an object $Y\in\mathcal{S}$ if and only if $X,Z\in\mathcal{S}$. A Serre subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{C}$ is *localizing* if it is closed under taking direct limits. Equivalently, the inclusion functor $i:\mathcal{S}\to\mathcal{C}$ admits the right adjoint $t=t_{\mathcal{S}}:\mathcal{C}\to\mathcal{S}$ which takes every object $X\in\mathcal{C}$ to the maximal subobject $t(X)$ of X belonging to $\mathcal{S}$. The functor t we call the *torsion functor*. An object C of $\mathcal{C}$ is said to be $\mathcal{S}$*-torsionfree* if $t(C)=0$. Given a localizing subcategory $\mathcal{S}$ of $\mathcal{C}$ the *quotient category* $\mathcal{C}/\mathcal{S}$ consists of $C\in\mathcal{C}$ such that $t(C)=t^1(C)=0$. The objects from $\mathcal{C}/\mathcal{S}$ we call $\mathcal{S}$*-closed objects*. Given $C\in\mathcal{C}$ there exists a canonical exact sequence

$$0\to A'\to C\xrightarrow{\lambda_C}C_{\mathcal{S}}\to A''\to 0$$

with $A'=t(C)$, $A''\in\mathcal{S}$, and where $C_{\mathcal{S}}\in\mathcal{C}/\mathcal{S}$ is the maximal essential extension of $\widetilde{C}=C/t(C)$ such that $C_{\mathcal{S}}/\widetilde{C}\in\mathcal{S}$. The object $C_{\mathcal{S}}$ is uniquely defined up to a canonical isomorphism and is called the $\mathcal{S}$*-envelope* of C. Moreover, the inclusion

functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ has the left adjoint *localizing functor* $(-)_\mathcal{S} : \mathcal{C} \to \mathcal{C}/\mathcal{S}$, which is also exact. It takes each $C \in \mathcal{C}$ to $C_\mathcal{S} \in \mathcal{C}/\mathcal{S}$. Then,

$$\operatorname{Hom}_\mathcal{C}(X,Y) \cong \operatorname{Hom}_{\mathcal{C}/\mathcal{S}}(X_\mathcal{S},Y)$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}/\mathcal{S}$.

If $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck categories, $q : \mathcal{C} \to \mathcal{D}$ is an exact functor, and a functor $s : \mathcal{D} \to \mathcal{C}$ is fully faithful and right adjoint to q, then $\mathcal{S} := \operatorname{Ker} q$ is a localizing subcategory and there exists an equivalence $\mathcal{C}/\mathcal{S} \overset{H}{\cong} \mathcal{D}$ such that $H \circ (-)_\mathcal{S} = q$. We shall refer to the pair (q,s) as the *localization pair*.

The following result is an example of the localization pair.

Proposition 2.1. *(cf.* [6, §III.5; Prop. VI.3]*) Let X be a scheme, let U be an open subset of X such that the canonical injection $j : U \to X$ is a quasi-compact map. Then $j_*(\mathcal{G})$ is a quasi-coherent $\mathcal{O}_X$-module for any quasi-coherent $\mathcal{O}_X|_U$-module $\mathcal{G}$ and the pair of adjoint functors (j^*, j_*) is a localization pair. That is the category of quasi-coherent $\mathcal{O}_X|_U$-modules* $\operatorname{Qcoh}(U)$ *is equivalent to* $\operatorname{Qcoh}(X)/\mathcal{S}$*, where $\mathcal{S} = \operatorname{Ker} j^*$. Moreover, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ belongs to the localizing subcategory $\mathcal{S}$ if and only if $\mathsf{supp}_X(\mathcal{F}) = \{P \in X \mid \mathcal{F}_P \neq 0\} \subseteq Z = X \setminus U$. Also, for any $\mathcal{F} \in \operatorname{Qcoh}(X)$ we have $t_\mathcal{S}(\mathcal{F}) = \mathcal{H}^0_Z(\mathcal{F})$, where $\mathcal{H}^0_Z(\mathcal{F})$ stands for the subsheaf of $\mathcal{F}$ with supports in Z.*

Proof. The fact that $j_*(\mathcal{G})$ is a quasi-coherent $\mathcal{O}_X$-module follows from [11, I.6.9.2]. The functor $j^* : \mathcal{F} \mapsto \mathcal{F}|_U$ is clearly exact, $j_*(\mathcal{G})|_U = j^* j_*(\mathcal{G}) = \mathcal{G}$ by [11, I.6.9.2]. It follows that j_* is fully faithful, and hence (j^*, j_*) is a localization pair.

The fact that $\mathcal{F} \in \mathcal{S}$ if and only if $\mathsf{supp}_X(\mathcal{F}) \subseteq Z$ is obvious. Finally, by [12, Ex. II.1.20] we have an exact sequence

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \xrightarrow{\rho_\mathcal{F}} j_* j^*(\mathcal{F}).$$

Since the morphism $\rho_\mathcal{F}$ can be regarded as an $\mathcal{S}$-envelope for $\mathcal{F}$, we see that $\operatorname{Ker} \rho_\mathcal{F} = t_\mathcal{S}(\mathcal{F}) = \mathcal{H}^0_Z(\mathcal{F})$. □

Given a subcategory $\mathcal{X}$ of a Grothendieck category $\mathcal{C}$, we denote by $\sqrt{\mathcal{X}}$ the smallest localizing subcategory of $\mathcal{C}$ containing $\mathcal{X}$. To describe $\sqrt{\mathcal{X}}$ intrinsically, we need the notion of a subquotient.

Definition. Given objects $A, B \in \mathcal{C}$, we say that A is a *subquotient* of B, or $A \prec B$, if there is a filtration of B by subobjects $B = B_0 \geqslant B_1 \geqslant B_2 \geqslant 0$ such that $A \cong B_1/B_2$. In other words, A is isomorphic to a subobject of a quotient object of B.

Given a subcategory $\mathcal{X}$ of $\mathcal{C}$, we denote by $\langle \mathcal{X} \rangle$ the full subcategory of subquotients of objects from $\mathcal{X}$. Clearly, $\langle \mathcal{X} \rangle = \langle \langle \mathcal{X} \rangle \rangle$, for the relation $A \prec B$ is transitive, and $\mathcal{X} = \langle \mathcal{X} \rangle$ if and only if $\mathcal{X}$ is closed under subobjects and quotient objects. If $\mathcal{X}$ is closed under direct sums then so is $\langle \mathcal{X} \rangle$.

Proposition 2.2. *Given a subcategory $\mathcal{X}$ of a Grothendieck category $\mathcal{C}$, an object $X \in \sqrt{\mathcal{X}}$ if and only if there is a filtration*

$$X_0 \subset X_1 \subset \cdots \subset X_\beta \subset \cdots$$

such that $X = \bigcup_\beta X_\beta$, $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$ if γ is a limit ordinal, and $X_0, X_{\beta+1}/X_\beta \in \langle \mathcal{X}^\oplus \rangle$ with $\mathcal{X}^\oplus$ standing for the subcategory of $\mathcal{C}$ consisting of direct sums of objects in $\mathcal{X}$.

Proof. It is easy to see that every object having such a filtration belongs to $\sqrt{\mathcal{X}}$. It is enough to show that the full subcategory $\mathcal{S}$ of such objects is localizing. Let

$$X \rightarrowtail Y \overset{g}{\twoheadrightarrow} Z$$

be a short exact sequence with $X, Z \in \mathcal{S}$. Let $X_0 \subset \cdots \subset X_\beta \subset \cdots$ and $Z_0 \subset \cdots \subset Z_\alpha \subset \cdots$ be the corresponding filtrations. Put $Y_\alpha = g^{-1}(Z_\alpha)$. Then we have a short exact sequence for any α

$$X \rightarrowtail Y_\alpha \overset{g_\alpha}{\twoheadrightarrow} Z_\alpha$$

with $Y_{\alpha+1}/Y_\alpha \cong Z_{\alpha+1}/Z_\alpha$. We have the following filtration for Y:

$$X_0 \subset \cdots \subset X_\beta \subset \cdots \subset X = \bigcup_\beta X_\beta \subset Y_0 \subset \cdots \subset Y_\alpha \subset \cdots$$

It follows that $Y \in \mathcal{S}$. We see that $\mathcal{S}$ is closed under extensions.

Now let $Y \in \mathcal{S}$ with a filtration $Y_0 \subset \cdots \subset Y_\alpha \subset \cdots$. Set $X_\alpha = X \cap Y_\alpha$ and $Z_\alpha = Y_\alpha/X_\alpha$. We get filtrations $X_0 \subset \cdots \subset X_\alpha \subset \cdots$ and $Z_0 \subset \cdots \subset Z_\alpha \subset \cdots$ for X and Z respectively. Thus $X, Z \in \mathcal{S}$, and so $\mathcal{S}$ is a Serre subcategory. It is plainly closed under direct sums, hence it is localizing. $\square$

Corollary 2.3. *Let $\mathcal{X}$ be a subcategory in $\mathcal{C}$ closed under subobjects, quotient objects, and direct sums. An object $M \in \mathcal{C}$ is $\sqrt{\mathcal{X}}$-closed if and only if* $\operatorname{Hom}(X, M) = \operatorname{Ext}^1(X, M) = 0$ *for all $X \in \mathcal{X}$.*

Proof. Suppose $\operatorname{Hom}(X,M) = \operatorname{Ext}^1(X,M) = 0$ for all $X \in \mathcal{X}$. We have to check that $\operatorname{Hom}(Y,M) = \operatorname{Ext}^1(Y,M) = 0$ for all $Y \in \sqrt{\mathcal{X}}$. By Proposition 2.2 there is a filtration

$$Y_0 \subset Y_1 \subset \cdots \subset Y_\beta \subset \cdots$$

such that $Y = \bigcup_\beta Y_\beta$, $Y_\gamma = \bigcup_{\beta<\gamma} Y_\beta$ if γ is a limit ordinal, and $Y_0, Y_{\beta+1}/Y_\beta \in \langle \mathcal{X}^\oplus \rangle = \mathcal{X}$. One has an exact sequence for any β

$$\begin{aligned}&\operatorname{Hom}(Y_{\beta+1}/Y_\beta, M) \to \operatorname{Hom}(Y_{\beta+1}, M) \to \operatorname{Hom}(Y_\beta, M) \to \\ &\to \operatorname{Ext}^1(Y_{\beta+1}/Y_\beta, M) \to \operatorname{Ext}^1(Y_{\beta+1}, M) \to \operatorname{Ext}^1(Y_\beta, M).\end{aligned}$$

One sees that if $\operatorname{Hom}(Y_\beta,M) = \operatorname{Ext}^1(Y_\beta,M) = 0$ then $\operatorname{Hom}(Y_{\beta+1},M) = \operatorname{Ext}^1(Y_{\beta+1},M) = 0$, because $Y_{\beta+1}/Y_\beta \in \mathcal{X}$. Since $Y_0 \in \mathcal{X}$ it follows that $\operatorname{Hom}(Y_\beta,M) = \operatorname{Ext}^1(Y_\beta,M) = 0$ for all finite β.

Let γ be a limit ordinal and $\operatorname{Hom}(Y_\beta,M) = \operatorname{Ext}^1(Y_\beta,M) = 0$ for all $\beta < \gamma$. We have $\operatorname{Hom}(Y_\gamma, M) = \varprojlim_{\beta<\gamma} \operatorname{Hom}(Y_\beta, M) = 0$. Let us show that $\operatorname{Ext}^1(Y_\gamma, M) = 0$. To see this we must prove that every short exact sequence

$$M \rightarrowtail N \twoheadrightarrow Y_\gamma$$

is split. One can construct a commutative diagram

$$\begin{tikzcd}
E_\beta: & M \arrow[r, tail] \arrow[d, equal] & N_\beta \arrow[r, two heads, shift left, "p_\beta"] \arrow[d, tail] & Y_\beta \arrow[l, shift left, "\exists \varkappa_\beta"] \arrow[d, tail] \\
E_\gamma: & M \arrow[r, tail] & N \arrow[r, two heads, "p"] & Y_\gamma
\end{tikzcd}$$

with $N_\beta = p^{-1}(Y_\beta)$. Clearly, $E_\gamma = \bigcup_\beta E_\beta$. Since the upper row splits, there exists a morphism $\varkappa_\beta$ such that $p_\beta \varkappa_\beta = 1$. Consider the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, equal] & N_\beta \arrow[r, two heads, "p_\beta"] \arrow[d, tail, "u_\beta"'] & Y_\beta \arrow[d, tail, "v_\beta"] \\ M \arrow[r, tail] & N_{\beta+1} \arrow[r, two heads, "p_{\beta+1}"] & Y_{\beta+1} \end{tikzcd}$$

We want to check that $\varkappa_{\beta+1} v_\beta = u_\beta \varkappa_\beta$. Since the right square is cartesian and $p_{\beta+1}\varkappa_{\beta+1} v_\beta = v_\beta$, there exists a unique morphism $\tau : Y_\beta \to N_\beta$ such that $p_\beta \tau = 1$ and $u_\beta \tau = \varkappa_{\beta+1} v_\beta$. We claim that $\tau = \varkappa_\beta$. Indeed, $p_\beta(\tau - \varkappa_\beta) = 0$ and hence $\tau - \varkappa_\beta$ factors through M. The latter is possible only if $\tau - \varkappa_\beta = 0$, for $\operatorname{Hom}(Y_\beta, M) = 0$ by assumption. Therefore $\tau = \varkappa_\beta$. It follows that the family of morphisms $\varkappa_\beta : Y_\beta \to N_\beta$ is directed and then $p \circ (\varinjlim \varkappa_\beta) = (\varinjlim p_\beta) \circ (\varinjlim \varkappa_\beta) = \varinjlim (p_\beta \varkappa_\beta) = 1$. Thus p is split. $\square$

Recall that the *injective spectrum* or the *Gabriel spectrum* $\operatorname{Sp} \mathcal{C}$ of a Grothendieck category $\mathcal{C}$ is the set of isomorphism classes of injective objects in $\mathcal{C}$. It plays an important role in our analysis. Given a subcategory $\mathcal{X}$ in $\mathcal{C}$ we denote by

$$(\mathcal{X}) = \{E \in \operatorname{Sp} \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(X, E) \neq 0 \text{ for some } X \in \mathcal{X}\}.$$

Using Proposition 2.1 and the fact that the functor $\operatorname{Hom}(-, E)$, $E \in \operatorname{Sp} \mathcal{C}$, is exact, we have $(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} (X) = (\sqrt{\mathcal{X}})$.

Proposition 2.4. *The collection of subsets of* $\operatorname{Sp} \mathcal{C}$*,*

$$\{(\mathcal{S}) \mid \mathcal{S} \subset \mathcal{C} \text{ is a localizing subcategory}\},$$

satisfies the axioms for the open sets of a topology on the injective spectrum $\operatorname{Sp} \mathcal{C}$*. This topological space will be denoted by* $\operatorname{Sp}_{gab} \mathcal{C}$*. Moreover, the map*

$$\mathcal{S} \longmapsto (\mathcal{S}) \tag{2.1}$$

is an inclusion-preserving bijection between the localizing subcategories $\mathcal{S}$ *of* $\mathcal{C}$ *and the open subsets of* $\operatorname{Sp}_{gab} \mathcal{C}$*.*

Proof. First note that $(0) = \emptyset$ and $(\mathcal{C}) = \operatorname{Sp} \mathcal{C}$. We have $(\mathcal{S}_1) \cap (\mathcal{S}_2) = (\mathcal{S}_1 \cap \mathcal{S}_2)$ because every $E \in \operatorname{Sp} \mathcal{C}$ is uniform and $0 \neq t_{\mathcal{S}_1}(E) \cap t_{\mathcal{S}_2}(E) \in \mathcal{S}_1 \cap \mathcal{S}_2$ whenever $E \in (\mathcal{S}_1) \cap (\mathcal{S}_2)$. Also, $\cup_{i \in I} (\mathcal{S}_i) = (\cup_{i \in I} \mathcal{S}_i) = (\sqrt{\cup_{i \in I} \mathcal{S}_i})$.

The map (2.1) is plainly bijective, because every localizing subcategory $\mathcal{S}$ consists precisely of those objects X such that $\operatorname{Hom}(X, E) = 0$ for all $E \in \operatorname{Sp} \mathcal{C} \setminus (\mathcal{S})$. $\square$

Given a localizing subcategory $\mathcal{S}$ in $\mathcal{C}$, the injective spectrum $\operatorname{Sp}_{gab}(\mathcal{C}/\mathcal{S})$ can be considered as the closed subset $\operatorname{Sp}_{gab} \mathcal{C} \setminus (\mathcal{S})$. Moreover, the inclusion

$$\operatorname{Sp}_{gab}(\mathcal{C}/\mathcal{S}) \hookrightarrow \operatorname{Sp}_{gab} \mathcal{C}$$

is a closed map. Indeed, if U is a closed subset in $\operatorname{Sp}_{gab}(\mathcal{C}/\mathcal{S})$ then there is a unique localizing subcategory $\mathcal{T}$ in $\mathcal{C}/\mathcal{S}$ such that $U = \operatorname{Sp}_{gab}(\mathcal{C}/\mathcal{S}) \setminus (\mathcal{T})$. By [7, 1.7] there is a unique localizing subcategory $\mathcal{P}$ in $\mathcal{C}$ containing $\mathcal{S}$ such that $\mathcal{C}/\mathcal{P}$ is equivalent to $(\mathcal{C}/\mathcal{S})/\mathcal{T}$. It follows that $U = \operatorname{Sp}_{gab} \mathcal{C} \setminus (\mathcal{P})$, hence U is closed in $\operatorname{Sp}_{gab} \mathcal{C}$.

On the other hand, let $\mathcal{Q}$ be a localizing subcategory of $\mathcal{C}$. Let us show that $O := (\mathcal{Q}) \cap \operatorname{Sp}_{gab}(\mathcal{C}/\mathcal{S})$ is an open subset in $\operatorname{Sp}_{gab}(\mathcal{C}/\mathcal{S})$.

Lemma 2.5. *Let $\widehat{\mathcal{Q}}$ denote the full subcategory of objects of the form $X_{\mathcal{S}}$ with $X \in \mathcal{Q}$. Then $\widehat{\mathcal{Q}}$ is closed under direct sums, subobjects, quotient objects in $\mathcal{C}/\mathcal{S}$ and $O = (\sqrt{\widehat{\mathcal{Q}}})$. Moreover, if $\mathcal{T}$ is the unique localizing subcategory of $\mathcal{C}$ containing $\mathcal{S}$ such that $\mathcal{C}/\mathcal{T} \cong (\mathcal{C}/\mathcal{S})/\sqrt{\widehat{\mathcal{Q}}}$ then the following relation is true:*

$$\mathcal{T} = \sqrt{(\mathcal{Q} \cup \mathcal{S})},$$

that is $\mathcal{T}$ is the smallest localizing subcategory containing $\mathcal{Q}$ and $\mathcal{S}$. We shall also refer to $\mathcal{T}$ as the join *of $\mathcal{Q}$ and $\mathcal{S}$.*

Proof. First let us prove that $\widehat{\mathcal{Q}}$ is closed under direct sums, subobjects, quotient objects in $\mathcal{C}/\mathcal{S}$. It is plainly closed under direct sums. Let Y be a subobject of $X_{\mathcal{S}}$, $X \in \mathcal{Q}$, and let $\lambda_X : X \to X_{\mathcal{S}}$ be the $\mathcal{S}$-envelope for X. Then $W = \lambda_X^{-1}(Y)$ is a subobject of X, hence it belongs to $\mathcal{Q}$, and $Y = W_{\mathcal{S}}$. If Z is a $\mathcal{C}/\mathcal{S}$-quotient of $X_{\mathcal{S}}$ and $\pi : X_{\mathcal{S}} \twoheadrightarrow Z$ is the canonical projection, then $Z = V_{\mathcal{S}}$ with $V = X/\mathrm{Ker}(\pi\lambda_X) \in \mathcal{Q}$. So $\widehat{\mathcal{Q}}$ is also closed under subobjects and quotient objects in $\mathcal{C}/\mathcal{S}$.

It follows that $\langle \widehat{\mathcal{Q}}^{\oplus} \rangle = \widehat{\mathcal{Q}}$ and $(\widehat{\mathcal{Q}}) = (\sqrt{\widehat{\mathcal{Q}}})$. On the other hand, $O = (\widehat{\mathcal{Q}})$ as one easily sees. Thus O is open in $\mathsf{Sp}_{gab}(\mathcal{C}/\mathcal{S})$.

Clearly,

$$(\mathcal{T}) = (\mathcal{Q}) \cup (\mathcal{S}) = (\mathcal{Q} \cup \mathcal{S}) = (\sqrt{(\mathcal{Q} \cup \mathcal{S})}).$$

By Proposition 2.4 $\mathcal{T} = \sqrt{(\mathcal{Q} \cup \mathcal{S})}$. □

We summarize the above arguments as follows.

Proposition 2.6. *Given a localizing subcategory $\mathcal{S}$ in $\mathcal{C}$, the topology on $\mathsf{Sp}_{gab}(\mathcal{C}/\mathcal{S})$ coincides with the subspace topology induced by $\mathsf{Sp}_{gab}\,\mathcal{C}$.*

3. Finite localizations of Grothendieck categories

In this paper we are mostly interested in finite localizations of a Grothendieck category $\mathcal{C}$. For this we should assume some finiteness conditions for $\mathcal{C}$.

Recall that an object X of a Grothendieck category $\mathcal{C}$ is *finitely generated* if whenever there are subobjects $X_i \subseteq X$ with $i \in I$ satisfying $X = \sum_{i \in I} X_i$, then there is a finite subset $J \subset I$ such that $X = \sum_{i \in J} X_i$. The subcategory of finitely generated objects is denoted by $\mathrm{fg}\,\mathcal{C}$. A finitely generated object X is said to be *finitely presented* if every epimorphism $\gamma : Y \to X$ with $Y \in \mathrm{fg}\,\mathcal{C}$ has the finitely generated kernel $\mathrm{Ker}\,\gamma$. By $\mathrm{fp}\,\mathcal{C}$ we denote the subcategory consisting of finitely presented objects. The category $\mathcal{C}$ is *locally finitely presented* if every object $C \in \mathcal{C}$ is a direct limit $C = \varinjlim C_i$ of finitely presented objects C_i, or equivalently, $\mathcal{C}$ possesses a family of finitely presented generators. In such a category, every finitely generated object $A \in \mathcal{C}$ admits an epimorphism $\eta : B \to A$ from a finitely presented object B. Finally, we refer to a finitely presented object $X \in \mathcal{C}$ as *coherent* if every finitely generated subobject of X is finitely presented. The corresponding subcategory of coherent objects will be denoted by $\mathrm{coh}\,\mathcal{C}$. A locally finitely presented category $\mathcal{C}$ is *locally coherent* if $\mathrm{coh}\,\mathcal{C} = \mathrm{fp}\,\mathcal{C}$. Obviously, a locally finitely presented category $\mathcal{C}$ is locally coherent if and only if $\mathrm{coh}\,\mathcal{C}$ is an abelian category.

In [5] it is shown that the category of quasi-coherent sheaves $\mathrm{Qcoh}(X)$ over a scheme X is a locally λ-presentable category, for λ a certain regular cardinal. However for some nice schemes which are in practise the most used for algebraic geometers like quasi-compact and quasi-separated there are enough finitely presented generators for $\mathrm{Qcoh}(X)$.

Proposition 3.1. *Let X be a quasi-compact and quasi-separated scheme. Then* $\mathrm{Qcoh}(X)$ *is a locally finitely presented Grothendieck category. An object* $\mathcal{F} \in \mathrm{fp}(\mathrm{Qcoh}(X))$ *if and only if it is locally finitely presented.*

Proof. By [11, I.6.9.12] every quasi-coherent sheaf is a direct limit of locally finitely presented sheaves. It follows from [21, Prop. 75] that the locally finitely presented sheaves are precisely the finitely presented objects in $\mathrm{Qcoh}(X)$. $\square$

Recall that a localizing subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{C}$ is *of finite type* (respectively *of strictly finite type*) if the functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ preserves directed sums (respectively direct limits). If $\mathcal{C}$ is a locally finitely generated (respectively, locally finitely presented) Grothendieck category and $\mathcal{S}$ is of finite type (respectively, of strictly finite type), then $\mathcal{C}/\mathcal{S}$ is a locally finitely generated (respectively, locally finitely presented) Grothendieck category and

$$\mathrm{fg}(\mathcal{C}/\mathcal{S}) = \{C_{\mathcal{S}} \mid C \in \mathrm{fg}\,\mathcal{C}\} \quad \text{(respectively } \mathrm{fp}(\mathcal{C}/\mathcal{S}) = \{C_{\mathcal{S}} \mid C \in \mathrm{fp}\,\mathcal{C}\}\text{)}.$$

If $\mathcal{C}$ is a locally coherent Grothendieck category then $\mathcal{S}$ is of finite type if and only if it is of strictly finite type (see, e.g., [7, 5.14]). In this case $\mathcal{C}/\mathcal{S}$ is locally coherent.

The following proposition says that localizing subcategories of finite type in a locally finitely presented Grothendieck category $\mathcal{C}$ are completely determined by finitely presented torsion objects (cf. [13, 19]).

Proposition 3.2. *Let $\mathcal{S}$ be a localizing subcategory of finite type in a locally finitely presented Grothendieck category $\mathcal{C}$. Then the following relation is true:*

$$\mathcal{S} = \sqrt{}(\mathrm{fp}\,\mathcal{C} \cap \mathcal{S}).$$

Proof. Obviously, $\sqrt{}(\mathrm{fp}\,\mathcal{C} \cap \mathcal{S}) \subset \mathcal{S}$. Let $X \in \mathcal{S}$ and let Y be a finitely generated subobject of X. There is an epimorphism $\eta : Z \to Y$ with $Z \in \mathrm{fp}\,\mathcal{C}$. By [7, 5.8] there is a finitely generated subobject $W \subset \mathrm{Ker}\,\eta$ such that $Z/W \in \mathcal{S}$. It follows that $Z/W \in \mathrm{fp}\,\mathcal{C} \cap \mathcal{S}$ and Y is an epimorphic image of Z/W. Since X is a direct union of finitely generated torsion subobjects, we see that X is an epimorphic image of some $\oplus_{i \in I} S_i$ with each $S_i \in \mathrm{fp}\,\mathcal{C} \cap \mathcal{S}$. Therefore $\mathcal{S} \subset \sqrt{}(\mathrm{fp}\,\mathcal{C} \cap \mathcal{S})$. $\square$

Lemma 3.3. *Let $\mathcal{Q}$ and $\mathcal{S}$ be two localizing subcategories in a Grothendieck category $\mathcal{C}$. If $X \in \mathcal{C}$ is both $\mathcal{Q}$-closed and $\mathcal{S}$-closed, then it is $\mathcal{T} = \sqrt{}(\mathcal{Q} \cup \mathcal{S})$-closed.*

Proof. By Lemma 2.5 $\mathcal{C}/\mathcal{T} \cong (\mathcal{C}/\mathcal{S})/\sqrt{}\widehat{\mathcal{Q}}$, where $\widehat{\mathcal{Q}} = \{C_{\mathcal{S}} \in \mathcal{C}/\mathcal{S} \mid C \in \mathcal{Q}\}$ and $\widehat{\mathcal{Q}}$ is closed under direct sums, subobjects, quotient objects in $\mathcal{C}/\mathcal{S}$. To show that $X = X_{\mathcal{S}}$ is a $\mathcal{T}$-closed object it is enough to check that X is $\sqrt{}\widehat{\mathcal{Q}}$-closed in $\mathcal{C}/\mathcal{S}$. Obviously $\mathrm{Hom}_{\mathcal{C}/\mathcal{S}}(A, X) = 0$ for all $A \in \widehat{\mathcal{Q}}$.

Consider a short exact sequence in $\mathcal{C}/\mathcal{S}$

$$E: \quad X \rightarrowtail Y \overset{p}{\twoheadrightarrow} C_{\mathcal{S}}$$

with $C \in \mathcal{Q}$. One can construct a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} E' : & X \arrow[r, tail] \arrow[d, equal] & Y' \arrow[r, "p'"] \arrow[d] & C \arrow[d, "\lambda_C"] \\ E : & X \arrow[r, tail] & Y \arrow[r, "p"] & C_{\mathcal{S}} \end{tikzcd}$$

with the right square cartesian. Let $C' = \mathrm{Im}_{\mathcal{C}}\, p'$ then $C' \in \mathcal{Q}$, $C'_{\mathcal{S}} = C_{\mathcal{S}}$ and the short exact sequence

$$E'' : \quad X \rightarrowtail Y' \xrightarrow{p'} \!\!\!\!\rightarrow C'$$

splits, because $\mathrm{Ext}^1_{\mathcal{C}}(C', X) = 0$. It follows that E splits for $E = E'_{\mathcal{S}} = E''_{\mathcal{S}}$. Therefore X is $\sqrt{\widehat{\mathcal{Q}}}$-closed by Corollary 2.3. $\square$

Below we shall need the following

Lemma 3.4. *Given a family of localizing subcategories of finite type $\{\mathcal{S}_i\}_{i \in I}$ in a locally finitely presented Grothendieck category $\mathcal{C}$, their join $\mathcal{T} = \sqrt{(\cup_{i \in I} \mathcal{S}_i)}$ is a localizing subcategory of finite type.*

Proof. Let us first consider the case when I is finite. By induction it is enough to show that the join $\mathcal{T} = \sqrt{(\mathcal{Q} \cup \mathcal{S})}$ of two localizing subcategories of finite type $\mathcal{Q}$ and $\mathcal{S}$ is of finite type. We have to check that the inclusion functor $\mathcal{C}/\mathcal{T} \to \mathcal{C}$ respects directed sums. It is plainly enough to verify that $X = \sum_{\mathcal{C}} X_\alpha$ is a $\mathcal{T}$-closed object whenever each X_α is $\mathcal{T}$-closed. Since $\mathcal{Q}$ and $\mathcal{S}$ are of finite type, X is both $\mathcal{Q}$-closed and $\mathcal{S}$-closed. It follows from Lemma 3.3 that X is $\mathcal{T}$-closed. Therefore $\mathcal{T}$ is of finite type.

Now let $\{\mathcal{S}_i\}_{i \in I}$ be an arbitrary set of localizing subcategories of finite type. Without loss of generality we may assume that I is a directed set and $\mathcal{S}_i \subset \mathcal{S}_j$ for $i \le j$. Indeed, given a finite subset $J \subset I$ we denote by $\mathcal{S}_J$ the localizing subcategory of finite type $\sqrt{(\cup_{j \in J} \mathcal{S}_j)}$. Then the set R of all finite subsets J of I is plainly directed, $\mathcal{S}_J \subset \mathcal{S}_{J'}$ for any $J \subset J'$, and $\mathcal{T} = \sqrt{(\cup_{J \in R} \mathcal{S}_J)}$.

Let $\mathcal{X}$ denote the full subcategory of $\mathcal{C}$ of those objects which can be presented as directed sums $\sum X_\alpha$ with each X_α belonging to $\cup_{i \in I} \mathcal{S}_i$. Since I is a directed set and $\mathcal{S}_i \subset \mathcal{S}_j$ for $i \le j$, it follows that a direct sum $X = \oplus_{\gamma \in \Gamma} X_\gamma$ with each X_γ belonging to $\cup_{i \in I} \mathcal{S}_i$. is in $\mathcal{X}$. Indeed, $X = \sum X_S$ with S running through all finite subsets of Γ and $X_S = \oplus_{\gamma \in S} X_\gamma \in \cup_{i \in I} \mathcal{S}_i$. Therefore if $\{X_\beta\}_{\beta \in B}$ is a family of subobjects of an object X and each X_β belongs to $\cup_{i \in I} \mathcal{S}_i$, then the direct union $\sum X_\beta$ belongs to $\mathcal{X}$.

The subcategory $\mathcal{X}$ is closed under subobjects and quotient objects. Indeed, let $X = \sum X_\alpha$ with each X_α belonging to $\cup_{i \in I} \mathcal{S}_i$. Consider a short exact sequence

$$Y \rightarrowtail X \twoheadrightarrow Z.$$

We set $Y_\alpha = Y \cap X_\alpha$ and $Z_\alpha = X_\alpha / Y_\alpha \subset Z$. Then both Y_α and Z_α are in $\cup_{i \in I} \mathcal{S}_i$, $Y = Y \cap (\sum X_\alpha) = \sum Y \cap X_\alpha = \sum Y_\alpha$ and $Z = \sum Z_\alpha$. So $Y, Z \in \mathcal{X}$.

Clearly, $\mathcal{X}$ is closed under directed sums, in particular direct sums, hence $\mathcal{X} = \langle \mathcal{X}^{\oplus} \rangle$ and $\mathcal{T} = \sqrt{\mathcal{X}}$. If we show that every direct limit $C = \varinjlim C_\delta$ of $\mathcal{T}$-closed objects C_δ has no $\mathcal{T}$-torsion, it will follow from [7, 5.8] that $\mathcal{T}$ is of finite type.

Using Proposition 2.2, it is enough to check that $\mathrm{Hom}_{\mathcal{C}}(X, C) = 0$ for any object $X \in \mathcal{X}$. Let Y be a finitely generated subobject in X. There is an index $i_0 \in I$ such that $Y \in \mathcal{S}_{i_0}$ and an epimorphism $\eta : Z \twoheadrightarrow Y$ with $Z \in \mathrm{fp}\,\mathcal{C}$. By [7, 5.8] there exists a finitely generated subobject W of $\mathrm{Ker}\,\eta$ such that $Z/W \in \mathcal{S}_{i_0}$. Since $Z/W \in \mathrm{fp}\,\mathcal{C}$ then $\mathrm{Hom}(Z/W, C) = \varinjlim \mathrm{Hom}(Z/W, C_\delta) = 0$. We see that $\mathrm{Hom}(Y, C) = 0$, and hence $\mathrm{Hom}(X, C) = 0$. $\square$

Given a localizing subcategory of finite type $\mathcal{S}$ in $\mathcal{C}$, we denote by

$$O(\mathcal{S}) = \{E \in \mathsf{Sp}\,\mathcal{C} \mid t_{\mathcal{S}}(E) \neq 0\}.$$

The next result has been obtained by Herzog [13] and Krause [19] for locally coherent Grothendieck categories and by Garkusha-Prest [10] for the category of modules $\operatorname{Mod} R$ over a commutative ring R.

Theorem 3.5. *Suppose $\mathcal{C}$ is a locally finitely presented Grothendieck category. The collection of subsets of $\operatorname{Sp}\mathcal{C}$,*

$$\{O(\mathcal{S}) \mid \mathcal{S} \subset \mathcal{C} \text{ is a localizing subcategory of finite type}\},$$

satisfies the axioms for the open sets of a topology on the injective spectrum $\operatorname{Sp}\mathcal{C}$. This topological space will be denoted by $\operatorname{Sp}_{fl}\mathcal{C}$ and this topology will be referred to as the fl-topology *("fl" for finite localizations). Moreover, the map*

$$\mathcal{S} \longmapsto O(\mathcal{S}) \tag{3.1}$$

is an inclusion-preserving bijection between the localizing subcategories $\mathcal{S}$ of finite type in $\mathcal{C}$ and the open subsets of $\operatorname{Sp}_{fl}\mathcal{C}$.

Proof. First note that $O(\mathcal{S}) = (\mathcal{S})$, $O(0) = \emptyset$ and $O(\mathcal{C}) = \operatorname{Sp}\mathcal{C}$. We have $O(\mathcal{S}_1) \cap O(\mathcal{S}_2) = (\mathcal{S}_1 \cap \mathcal{S}_2)$ by Proposition 2.4. We claim that $\mathcal{S}_1 \cap \mathcal{S}_2$ is of finite type, whence $O(\mathcal{S}_1) \cap O(\mathcal{S}_2) = O(\mathcal{S}_1 \cap \mathcal{S}_2)$. Indeed, let us consider a morphism $f : X \to S$ from a finitely presented object X to an object $S \in \mathcal{S}_1 \cap \mathcal{S}_2$. It follows from [7, 5.8] that there are finitely generated subobjects $X_1, X_2 \subseteq \operatorname{Ker} f$ such that $X/X_i \in \mathcal{S}_i$, $i = 0,1$. Then $X_1 + X_2$ is a finitely generated subobject of $\operatorname{Ker} f$ and $X/(X_1 + X_2) \in \mathcal{S}_1 \cap \mathcal{S}_2$. By [7, 5.8] $\mathcal{S}_1 \cap \mathcal{S}_2$ is of finite type.

By Lemma 3.4 $\sqrt{\cup_{i\in I}\mathcal{S}_i}$ is of finite type with each $\mathcal{S}_i$ of finite type. It follows from Proposition 2.4 that $\cup_{i\in I} O(\mathcal{S}_i) = O(\cup_{i\in I}\mathcal{S}_i) = O(\sqrt{\cup_{i\in I}\mathcal{S}_i})$.

It follows from Proposition 2.4 that the map (3.1) is bijective. □

Let $L_{\mathrm{loc}}(\mathcal{C})$ denote the lattice of localizing subcategories of $\mathcal{C}$, where, by definition,

$$\mathcal{S} \wedge \mathcal{Q} = \mathcal{S} \cap \mathcal{Q}, \quad \mathcal{S} \vee \mathcal{Q} = \sqrt{(\mathcal{S} \cup \mathcal{Q})}$$

for any $\mathcal{S}, \mathcal{Q} \in L_{\mathrm{loc}}(\mathcal{C})$. The proof of Theorem 3.5 shows that the subset of localizing subcategories of finite type in $L_{\mathrm{loc}}(\mathcal{C})$ is a sublattice. We shall denote it by $L_{\mathrm{f.loc}}(\mathcal{C})$.

Remark. If $\mathcal{C}$ is a locally coherent Grothendieck category, the topological space $\operatorname{Sp}_{fl}\mathcal{C}$ is also called in literature the *Ziegler spectrum* of $\mathcal{C}$. It arises from the Ziegler work on the model theory of modules [27]. According to the original Ziegler definition the points of the Ziegler spectrum of a ring R are the isomorphism classes of indecomposable pure-injective right R-modules. These can be identified with $\operatorname{Sp}(R\operatorname{mod}, \mathrm{Ab})$, where $(R\operatorname{mod}, \mathrm{Ab})$ is the locally coherent Grothendieck category consisting of additive covariant functors defined on the category of finitely presented left modules $R\operatorname{mod}$ with values in the category of abelian groups Ab. The closed subsets correspond to complete theories of modules. Later Herzog [13] and Krause [19] defined the Ziegler topology for arbitrary locally coherent Grothendieck categories.

Proposition 3.6. *Given a localizing subcategory of strictly finite type $\mathcal{S}$ in a locally finitely presented Grothendieck category $\mathcal{C}$, the topology on $\operatorname{Sp}_{fl}(\mathcal{C}/\mathcal{S})$ coincides with the subspace topology induced by $\operatorname{Sp}_{fl}\mathcal{C}$.*

Proof. By [7, 5.9] $\mathcal{C}/\mathcal{S}$ is a locally finitely presented Grothendieck category, and so the fl-topology on $\mathsf{Sp}(\mathcal{C}/\mathcal{S})$ makes sense. Let $O(\mathcal{P})$ be an open subset of $\mathsf{Sp}_{fl}(\mathcal{C}/\mathcal{S})$ with $\mathcal{P}$ a localizing subcategory of finite type of $\mathcal{C}/\mathcal{S}$. There is a unique localizing subcategory $\mathcal{T}$ of $\mathcal{C}$ such that $(\mathcal{C}/\mathcal{S})/\mathcal{P} \cong \mathcal{C}/\mathcal{T}$. We claim that $\mathcal{T}$ is of finite type.

It is plainly enough to verify that $X = \sum_{\mathcal{C}} X_\alpha$ is a $\mathcal{T}$-closed object whenever each X_α is $\mathcal{T}$-closed. Since $\mathcal{S}$ and $\mathcal{P}$ are of finite type in $\mathcal{C}$ and $\mathcal{C}/\mathcal{S}$ respectively, X is both $\mathcal{S}$-closed and $\mathcal{P}$-closed in $\mathcal{C}$ and $\mathcal{C}/\mathcal{S}$ respectively. It follows that X is $\mathcal{T}$-closed. Therefore $\mathcal{T}$ is of finite type and $O(\mathcal{P}) = \mathsf{Sp}_{fl}(\mathcal{C}/\mathcal{S}) \cap O(\mathcal{T})$.

Now let $\mathcal{Q}$ be a localizing subcategory of finite type in $\mathcal{C}$. We want to show that $\mathsf{Sp}_{fl}(\mathcal{C}/\mathcal{S}) \cap O(\mathcal{Q})$ is open in $\mathsf{Sp}_{fl}(\mathcal{C}/\mathcal{S})$. Let $\widehat{\mathcal{Q}} = \{X_{\mathcal{S}} \mid X \in \mathcal{Q}\}$, then $\widehat{\mathcal{Q}}$ is closed under direct sums, subobjects, quotient objects in $\mathcal{C}/\mathcal{S}$ (see the proof of Lemma 2.5) and $O(\widehat{\mathcal{Q}}) = O(\sqrt{\widehat{\mathcal{Q}}}) = \mathsf{Sp}_{fl}(\mathcal{C}/\mathcal{S}) \cap O(\mathcal{Q})$. We have to show that $\sqrt{\widehat{\mathcal{Q}}}$ is of finite type in $\mathcal{C}/\mathcal{S}$.

If we show that every direct limit $C = \varinjlim_{\mathcal{C}/\mathcal{S}} C_\delta$ of $\sqrt{\widehat{\mathcal{Q}}}$-closed objects C_δ has no $\sqrt{\widehat{\mathcal{Q}}}$-torsion, it will follow from [7, 5.8] that $\sqrt{\widehat{\mathcal{Q}}}$ is of finite type. Obviously, each C_δ is $\mathcal{Q}$-closed.

Using Proposition 2.2, it is enough to check that $\operatorname{Hom}_{\mathcal{C}}(X, C) = 0$ for any object $X \in \widehat{\mathcal{Q}}$. Since $\mathcal{S}$ is of strictly finite type, one has $C \cong \varinjlim_{\mathcal{C}} C_\delta$. Each C_δ is $\mathcal{Q}$-closed, and therefore $\varinjlim_{\mathcal{C}} C_\delta$ has no $\mathcal{Q}$-torsion by [7, 5.8] and the fact that $\mathcal{Q}$ is of finite type. There is an object $Y \in \mathcal{Q}$ such that $Y_{\mathcal{S}} = X$. Then $\operatorname{Hom}_{\mathcal{C}/\mathcal{S}}(X, C) \cong \operatorname{Hom}_{\mathcal{C}}(Y, \varinjlim_{\mathcal{C}} C_\delta) = 0$, as required. $\square$

4. The topological space $\mathsf{Sp}_{fl,\otimes}(X)$

In the preceding section we studied some general properties of finite localizations in locally finitely presented Grothendieck categories and their relation with the topological space $\mathsf{Sp}_{fl}\,\mathcal{C}$. In this section we introduce and study the topological space $\mathsf{Sp}_{fl,\otimes}(X)$ which is of particular importance in practice. If otherwise specified, X is supposed to be quasi-compact and quasi-separated.

Given a quasi-compact open subset $U \subset X$, we denote by $\mathcal{S}_U = \{\mathcal{F} \in \operatorname{Qcoh}(X) \mid \mathcal{F}|_U = 0\}$. It follows from [7, 5.9] and the fact that $\mathcal{F}|_U \in \operatorname{fp}(\operatorname{Qcoh}(U))$ whenever $\mathcal{F} \in \operatorname{fp}(\operatorname{Qcoh}(X))$ that $\mathcal{S}_U$ is of strictly finite type. Below we shall need the following

Lemma 4.1. *Let X be a quasi-compact and quasi-separated scheme and let U, V be quasi-compact open subsets. Then the following relation holds:*

$$\mathcal{S}_{U\cap V} = \sqrt{(\mathcal{S}_U \cup \mathcal{S}_V)}.$$

Proof. Clearly, $\mathcal{S}_{U\cap V}$ contains both $\mathcal{S}_U$ and $\mathcal{S}_V$ and so $\mathcal{S}_{U\cap V} \supset \sqrt{(\mathcal{S}_U \cup \mathcal{S}_V)}$. Let $\mathcal{F} \in \mathcal{S}_{U\cap V}$ and let $j : U \to X$ be the canonical inclusion. Then $j_* j^*(\mathcal{F}) \in \mathcal{S}_V$. One has the following exact sequence

$$0 \to t_{\mathcal{S}_U}(\mathcal{F}) \to \mathcal{F} \xrightarrow{\lambda_{\mathcal{F}}} j_* j^*(\mathcal{F}).$$

Since $t_{\mathcal{S}_U}(\mathcal{F}) \in \mathcal{S}_U$ and $\operatorname{Im}(\lambda_{\mathcal{F}}) \in \mathcal{S}_V$, we see that $\mathcal{F} \in \sqrt{(\mathcal{S}_U \cup \mathcal{S}_V)}$. $\square$

We denote by $\mathsf{Sp}_{fl}(X)$ the topological space $\mathsf{Sp}_{fl}(\operatorname{Qcoh}(X))$.

Corollary 4.2. *Let X be a quasi-compact and quasi-separated scheme and $X = U \cup V$ with U, V quasi-compact open subsets. Then the following relations hold:*

$$\mathsf{Sp}(X) = \mathsf{Sp}(U) \cup \mathsf{Sp}(V), \quad \mathsf{Sp}(U \cap V) = \mathsf{Sp}(U) \cap \mathsf{Sp}(V)$$
$$\mathsf{Sp}_{fl}(X) = \mathsf{Sp}_{fl}(U) \cup \mathsf{Sp}_{fl}(V), \quad \mathsf{Sp}_{fl}(U \cap V) = \mathsf{Sp}_{fl}(U) \cap \mathsf{Sp}_{fl}(V).$$

Proof. It follows from the fact that $\mathcal{S}_U \cap \mathcal{S}_V = 0$, Propositions 2.1, 2.4, 2.6, 3.6, Theorem 3.5, and Lemma 4.1. □

Let $L_{\mathrm{loc}}(X)$ (respectively, $L_{\mathrm{f.loc}}(X)$) denote the lattice $L_{\mathrm{loc}}(\mathrm{Qcoh}(X))$ (respectively, $L_{\mathrm{f.loc}}(\mathrm{Qcoh}(X))$). It follows from Proposition 2.4 and Theorem 3.5 that the map $L_{\mathrm{loc}}(X) \to L_{\mathrm{open}}(\mathsf{Sp}(X))$ (respectively, $L_{\mathrm{f.loc}}(X) \to L_{\mathrm{open}}(\mathsf{Sp}_{fl}(X))$) is a lattice isomorphism. Suppose $U \subset X$ is a quasi-compact open subset. Then the map

$$\alpha_{X,U} : L_{\mathrm{loc}}(X) \to L_{\mathrm{loc}}(U), \quad \mathcal{S} \mapsto \sqrt{(\widehat{\mathcal{S}}|_U)},$$

where $\widehat{\mathcal{S}}|_U = \{\mathcal{F}|_U = \mathcal{F}_{\mathcal{S}_U} \mid \mathcal{F} \in \mathcal{S}\}$, is a lattice map. If V is another quasi-compact subset of X such that $X = U \cup V$ then, obviously,

$$\alpha_{X,U\cap V} = \alpha_{U,U\cap V} \circ \alpha_{X,U} = \alpha_{V,U\cap V} \circ \alpha_{X,V}.$$

By the proof of Proposition 3.6 $\alpha_{X,U}(\mathcal{S}) \in L_{\mathrm{f.loc}}(U)$ for every $\mathcal{S} \in L_{\mathrm{f.loc}}(U)$. Thus we have a map

$$\alpha_{X,U} : L_{\mathrm{f.loc}}(X) \to L_{\mathrm{f.loc}}(U).$$

The notion of a pullback for lattices satisfying the obvious universal property is easily defined.

Lemma 4.3. *The commutative squares of lattices*

$$\begin{array}{ccc} L_{\mathrm{loc}}(X) & \xrightarrow{\alpha_{X,U}} & L_{\mathrm{loc}}(U) \\ {\scriptstyle \alpha_{X,V}}\downarrow & & \downarrow{\scriptstyle \alpha_{U,U\cap V}} \\ L_{\mathrm{loc}}(V) & \xrightarrow{\alpha_{V,U\cap V}} & L_{\mathrm{loc}}(U\cap V) \end{array}$$

and

$$\begin{array}{ccc} L_{\mathrm{f.loc}}(X) & \xrightarrow{\alpha_{X,U}} & L_{\mathrm{f.loc}}(U) \\ {\scriptstyle \alpha_{X,V}}\downarrow & & \downarrow{\scriptstyle \alpha_{U,U\cap V}} \\ L_{\mathrm{f.loc}}(V) & \xrightarrow{\alpha_{V,U\cap V}} & L_{\mathrm{f.loc}}(U\cap V) \end{array}$$

are pullback.

Proof. It is enough to observe that the commutative squares of lemma are isomorphic to the corresponding pullback squares of lattices of open sets

$$\begin{array}{ccc} L_{\mathrm{open}}(\mathsf{Sp}(X)) & \xrightarrow{\alpha_{X,U}} & L_{\mathrm{open}}(\mathsf{Sp}(U)) \\ {\scriptstyle \alpha_{X,V}}\downarrow & & \downarrow{\scriptstyle \alpha_{U,U\cap V}} \\ L_{\mathrm{open}}(\mathsf{Sp}(V)) & \xrightarrow{\alpha_{V,U\cap V}} & L_{\mathrm{open}}(\mathsf{Sp}(U\cap V)) \end{array}$$

and

$$\begin{array}{ccc} L_{\mathrm{open}}(\mathsf{Sp}_{fl}(X)) & \xrightarrow{\alpha_{X,U}} & L_{\mathrm{open}}(\mathsf{Sp}_{fl}(U)) \\ {\scriptstyle \alpha_{X,V}}\downarrow & & \downarrow{\scriptstyle \alpha_{U,U\cap V}} \\ L_{\mathrm{open}}(\mathsf{Sp}_{fl}(V)) & \xrightarrow{\alpha_{V,U\cap V}} & L_{\mathrm{open}}(\mathsf{Sp}_{fl}(U\cap V)) \end{array}$$

(see Proposition 2.4, Theorem 3.5, and Corollary 4.2). □

Recall that $\mathrm{Qcoh}(X)$ is monoidal with the tensor product $\otimes_X$ right exact and preserving direct limits (see [18, §II.2]).

Definition. A localizing subcategory $\mathcal{S}$ of $\mathrm{Qcoh}(X)$ is said to be *tensor* if $\mathcal{F}\otimes_X\mathcal{G}\in\mathcal{S}$ for every $\mathcal{F}\in\mathcal{S}$ and $\mathcal{G}\in\mathrm{Qcoh}(X)$.

Lemma 4.4. *A localizing subcategory of finite type $\mathcal{S}\subset\mathrm{Qcoh}(X)$ is tensor if and only if $\mathcal{F}\otimes_X\mathcal{G}\in\mathcal{S}$ for every $\mathcal{F}\in\mathcal{S}\cap\mathrm{fp}(\mathrm{Qcoh}(X))$ and $\mathcal{G}\in\mathrm{fp}(\mathrm{Qcoh}(X))$.*

Proof. It is enough to observe that every $\mathcal{F}\in\mathcal{S}$ is a quotient object of the direct sum of objects from $\mathcal{S}\cap\mathrm{fp}(\mathrm{Qcoh}(X))$ and that every object $\mathcal{G}\in\mathrm{Qcoh}(X)$ is a direct limit of finitely presented objects. □

Lemma 4.5. *Let $\mathcal{X}\subset\mathrm{Qcoh}(X)$ be a subcategory closed under direct sums, subobjects, quotient objects, and tensor products. Then $\sqrt{\mathcal{X}}$ is a tensor localizing subcategory.*

Proof. By Proposition 2.2 an object $\mathcal{F}\in\sqrt{\mathcal{X}}$ if and only if there is a filtration

$$\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_\beta\subset\cdots$$

such that $\mathcal{F}=\bigcup_\beta\mathcal{F}_\beta$, $\mathcal{F}_\gamma=\bigcup_{\beta<\gamma}\mathcal{F}_\beta$ if γ is a limit ordinal, and $\mathcal{F}_0,\mathcal{F}_{\beta+1}/\mathcal{F}_\beta\in\langle\mathcal{X}^{\oplus}\rangle=\mathcal{X}$.

We have $\mathcal{F}_0\otimes_X\mathcal{G}\in\mathcal{X}$ for any $\mathcal{G}\in\mathrm{Qcoh}(X)$. Suppose $\beta=\alpha+1$ and $\mathcal{F}_\alpha\otimes_X\mathcal{G}\in\sqrt{\mathcal{X}}$. One has an exact sequence

$$\mathcal{F}_\alpha\otimes_X\mathcal{G}\xrightarrow{f}\mathcal{F}_\beta\otimes_X\mathcal{G}\to(\mathcal{F}_\beta/\mathcal{F}_\alpha)\otimes_X\mathcal{G}\to 0.$$

Since $(\mathcal{F}_\beta/\mathcal{F}_\alpha)\otimes_X\mathcal{G}\in\mathcal{X}$ and $\mathrm{Im}\,f\in\sqrt{\mathcal{X}}$, we see that $\mathcal{F}_\beta\otimes_X\mathcal{G}\in\sqrt{\mathcal{X}}$. If γ is a limit ordinal and $\mathcal{F}_\beta\otimes_X\mathcal{G}\in\sqrt{\mathcal{X}}$ for all $\beta<\gamma$, then $\mathcal{F}_\gamma\otimes_X\mathcal{G}=\varinjlim_{\beta<\gamma}(\mathcal{F}_\beta\otimes_X\mathcal{G})\in\sqrt{\mathcal{X}}$. Therefore $\sqrt{\mathcal{X}}$ is tensor. □

The next statement is of great utility in this paper.

Reduction principle. *Let $\mathfrak{S}$ be the class of quasi-compact, quasi-separated schemes and let P be a property satisfied by some schemes from $\mathfrak{S}$. Assume in addition the following.*

(1) *P is true for affine schemes.*
(2) *If $X\in\mathfrak{S}$, $X=U\cup V$, where U,V are quasi-compact open subsets of X, and P holds for $U,V,U\cap V$ then it holds for X.*

Then P holds for all schemes from $\mathfrak{S}$.

Proof. See the proof of [20, 3.9.2.4] and [3, 3.3.1]. □

Lemma 4.6. *The join $\mathcal{T}=\sqrt{(\mathcal{S}\cup\mathcal{Q})}$ of two tensor localizing subcategories $\mathcal{S},\mathcal{Q}\subset\mathrm{Qcoh}(X)$ is tensor.*

Proof. We use the Reduction Principle to demonstrate the lemma. It is true for affine schemes, because every localizing subcategory is tensor in this case. Suppose $X=U\cup V$, where U,V are quasi-compact open subsets of X, and the assertion is true for $U,V,U\cap V$. We have to show that it is true for X itself.

We have the following relation:

$$\alpha_{X,U}(\mathcal{T})=(\sqrt{(\widehat{\mathcal{S}}|_U)})\vee(\sqrt{(\widehat{\mathcal{Q}}|_U)}).$$

Given $\mathcal{F}, \mathcal{G} \in \mathrm{Qcoh}(X)$ there is a canonical isomorphism (see [18, II.2.3.5])

$$(\mathcal{F}|_U) \otimes_U (\mathcal{G}|_U) \cong (\mathcal{F} \otimes_X \mathcal{G})|_U.$$

It follows that both $\widehat{\mathcal{S}}|_U$ and $\widehat{\mathcal{Q}}|_U$ are closed under tensor products. By Lemma 4.5 both $\sqrt{\widehat{\mathcal{S}}}|_U$ and $\sqrt{\widehat{\mathcal{Q}}}|_U$ are tensor. By assumption, the join of two tensor localizing subcategories in $\mathrm{Qcoh}(U)$ is tensor, and so $\alpha_{X,U}(\mathcal{T})$ is tensor. For the same reasons, $\alpha_{X,V}(\mathcal{T})$ is tensor. Obviously, $\mathcal{T}$ is tensor whenever so are $\alpha_{X,U}(\mathcal{T})$ and $\alpha_{X,V}(\mathcal{T})$. Therefore $\mathcal{T}$ is tensor as well and our assertion now follows from the Reduction Principle. $\square$

Given a tensor localizing subcategory of finite type $\mathcal{S}$ in $\mathrm{Qcoh}(X)$, we denote by

$$\mathcal{O}(\mathcal{S}) = \{\mathcal{E} \in \mathsf{Sp}(X) \mid t_{\mathcal{S}}(\mathcal{E}) \neq 0\}.$$

Theorem 4.7. *The collection of subsets of the injective spectrum* $\mathsf{Sp}(X)$*,*

$$\{\mathcal{O}(\mathcal{S}) \mid \mathcal{S} \subset \mathcal{C} \textit{ is a tensor localizing subcategory of finite type}\},$$

satisfies the axioms for the open sets of a topology on $\mathsf{Sp}(X)$*. This topological space will be denoted by* $\mathsf{Sp}_{fl,\otimes}(X)$ *and this topology will be referred to as the* tensor fl-topology*. Moreover, the map*

$$\mathcal{S} \longmapsto \mathcal{O}(\mathcal{S}) \tag{4.1}$$

is an inclusion-preserving bijection between the tensor localizing subcategories $\mathcal{S}$ *of finite type in* $\mathrm{Qcoh}(X)$ *and the open subsets of* $\mathsf{Sp}_{fl,\otimes}(X)$*.*

Proof. Obviously, the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$ of two tensor localizing subcategories of finite type is a tensor localizing subcategory of finite type, hence $\mathcal{O}(\mathcal{S}_1 \cap \mathcal{S}_2) = \mathcal{O}(\mathcal{S}_1) \cap \mathcal{O}(\mathcal{S}_2)$ by Theorem 3.5.

Now let us show that the join $\mathcal{T} = \sqrt{(\cup_{i\in I}\mathcal{S}_i)}$ of tensor localizing subcategories of finite type $\mathcal{S}_i$ is tensor. By induction and Lemma 4.6 $\mathcal{T}$ is tensor whenever I is finite. Now we may assume, without loss of generality, that I is a directed index set and $\mathcal{S}_i \subset \mathcal{S}_j$ for any $i \leq j$. By the proof of Lemma 3.4 $\mathcal{T} = \sqrt{\mathcal{X}}$ with $\mathcal{X}$ the full subcategory of $\mathrm{Qcoh}(X)$ of those objects which can be presented as directed sums $\sum F_\alpha$ with each F_α belonging to $\cup_{i\in I}\mathcal{S}_i$. Then $\mathcal{X}$ is closed under direct sums, subobjects and quotient objects. It is also closed under tensor product, because $\otimes_X$ commutes with direct limits. By Lemma 4.5 $\mathcal{T}$ is tensor and by Lemma 3.4 $\mathcal{T}$ is of finite type.

Theorem 3.5 implies that $\mathcal{O}(\sqrt{(\cup_{i\in I}\mathcal{S}_i)}) = \cup_{i\in I}\mathcal{O}(\mathcal{S}_i)$ and the map (4.1) is bijective. $\square$

We denote by $L_{\mathrm{f.loc},\otimes}(X)$ the lattice of tensor localizing subcategories of finite type in $\mathrm{Qcoh}(X)$.

Corollary 4.8. *The commutative square of lattices*

$$\begin{array}{ccc}
L_{\mathrm{f.loc},\otimes}(X) & \xrightarrow{\alpha_{X,U}} & L_{\mathrm{f.loc},\otimes}(U) \\
{\scriptstyle\alpha_{X,V}}\downarrow & & \downarrow{\scriptstyle\alpha_{U,U\cap V}} \\
L_{\mathrm{f.loc},\otimes}(V) & \xrightarrow{\alpha_{V,U\cap V}} & L_{\mathrm{f.loc},\otimes}(U\cap V)
\end{array}$$

is pullback.

Proof. The proof is similar to that of Lemma 4.3. $\square$

5. The classification theorem

Recall from [14] that a topological space is *spectral* if it is T_0, quasi-compact, if the quasi-compact open subsets are closed under finite intersections and form an open basis, and if every non-empty irreducible closed subset has a generic point. Given a spectral topological space, X, Hochster [14] endows the underlying set with a new, "dual", topology, denoted X^*, by taking as open sets those of the form $Y = \bigcup_{i\in\Omega} Y_i$ where Y_i has quasi-compact open complement $X\setminus Y_i$ for all $i\in\Omega$. Then X^* is spectral and $(X^*)^* = X$ (see [14, Prop. 8]).

As an example, the underlying topological space of a quasi-compact, quasi-separated scheme X is spectral. In this section we shall show that the tensor localizing subcategories of finite type in $\mathrm{Qcoh}(X)$ are in 1-1 correspondence with the open subsets of X^*. If otherwise specified, X is supposed to be a quasi-compact, quasi-separated scheme.

Given a quasi-compact open subset $D\subset X$, we denote by $\mathcal{S}_D = \{\mathcal{F}\in\mathrm{Qcoh}(X) \mid \mathcal{F}|_D = 0\}$.

Proposition 5.1. *Given an open subset $O = \cup_I O_i \subset X^*$, where each $D_i = X\setminus O_i$ is quasi-compact and open in X, the subcategory $\mathcal{S} = \{\mathcal{F}\in\mathrm{Qcoh}(X) \mid \mathsf{supp}_X(\mathcal{F})\subseteq O\}$ is a tensor localizing subcategory of finite type and $\mathcal{S} = \sqrt{(\cup_I \mathcal{S}_{D_i})}$.*

Proof. Given a short exact sequence in $\mathrm{Qcoh}(X)$

$$\mathcal{F}' \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{F}'',$$

one has $\mathsf{supp}_X(\mathcal{F}) = \mathsf{supp}_X(\mathcal{F}')\cup\mathsf{supp}_X(\mathcal{F}'')$. It follows that $\mathcal{S}$ is a Serre subcategory. It is also closed under direct sums, hence localizing, because $\mathsf{supp}_X(\oplus_I\mathcal{F}_i) = \cup_I\mathsf{supp}_X(\mathcal{F}_i)$.

We use the Reduction Principle to show that $\mathcal{S}$ is a tensor localizing subcategory of finite type and $\mathcal{S} = \sqrt{(\cup_I\mathcal{S}_{D_i})}$. It is the case for affine schemes (see [10, 2.2]). Suppose $X = U\cup V$, where U,V are quasi-compact open subsets of X, and the assertion is true for $U,V,U\cap V$. We have to show that it is true for X itself.

For any $\mathcal{F}\in\mathrm{Qcoh}(X)$ we have

$$\mathsf{supp}_X(\mathcal{F}) = \mathsf{supp}_U(\mathcal{F}|_U)\cup\mathsf{supp}_V(\mathcal{F}|_V).$$

Clearly, $O\cap U$ is open in U^* and $\mathsf{supp}_U(\mathcal{F}|_U)\subseteq O\cap U$ for any $\mathcal{F}\in\mathcal{S}$. We see that $\widehat{\mathcal{S}}|_U = \{\mathcal{F}|_U = \mathcal{F}_{\mathcal{S}_U} \mid \mathcal{F}\in\mathcal{S}\}$ is contained in $\mathcal{S}(U) = \{\mathcal{F}\in\mathrm{Qcoh}(U)\mid\mathsf{supp}_U(\mathcal{F})\subseteq O\cap U\}$. By assumption, $\mathcal{S}(U)$ is a tensor localizing subcategory of finite type in $\mathrm{Qcoh}(U)$ and $\mathcal{S}(U) = \sqrt{(\cup_I\mathcal{S}_{D_i\cap U})}$. We have $\mathcal{S}(U)\supset\sqrt{(\widehat{\mathcal{S}}|_U)}$. Similarly, $\mathcal{S}(V)\supset\sqrt{(\widehat{\mathcal{S}}|_V)}$ and $\mathcal{S}(V) = \sqrt{(\cup_I\mathcal{S}_{D_i\cap V})}$.

Since

$$\alpha_{U,U\cap V}(\mathcal{S}_{D_i\cap U}) = \alpha_{V,U\cap V}(\mathcal{S}_{D_i\cap V}) \overset{\text{Lem. 4.1}}{=} \mathcal{S}_{D_i\cap U\cap V}$$
$$\overset{\text{Prop. 2.1}}{=} \{\mathcal{F}\in\mathrm{Qcoh}(U\cap V)\mid\mathsf{supp}_{U\cap V}(\mathcal{F})\subseteq O_i\cap U\cap V\},$$

it follows that

$$\alpha_{U,U\cap V}(\mathcal{S}(U)) = \alpha_{V,U\cap V}(\mathcal{S}(V)) =$$
$$\mathcal{S}(U\cap V) = \{\mathcal{F}\in\mathrm{Qcoh}(U\cap V)\mid\mathsf{supp}_{U\cap V}(\mathcal{F})\subseteq O\cap U\cap V\} = \sqrt{(\cup_I\mathcal{S}_{D_i\cap U\cap V})}.$$

By Corollary 4.8 there is a unique tensor localizing subcategory of finite type $\mathcal{T}\in L_{\mathrm{f.loc},\otimes}(X)$ such that $\sqrt{(\widehat{\mathcal{T}}|_U)} = \mathcal{S}(U)$, $\sqrt{(\widehat{\mathcal{T}}|_V)} = \mathcal{S}(V)$ and $\mathcal{T} = \sqrt{(\cup_I\mathcal{S}_{D_i})}$. By

construction, $\mathsf{supp}_X(\mathcal{F}) \subseteq O$ for all $\mathcal{F} \in \mathcal{T}$, and hence $\mathcal{T} \subseteq \mathcal{S}$, $\sqrt{(\widehat{\mathcal{T}}|_U)} \subseteq \sqrt{(\widehat{\mathcal{S}}|_U)}$, $\sqrt{(\widehat{\mathcal{T}}|_V)} \subseteq \sqrt{(\widehat{\mathcal{S}}|_V)}$. Therefore $\mathcal{S}(V) = \sqrt{(\widehat{\mathcal{S}}|_V)}$ and $\mathcal{S}(V) = \sqrt{(\widehat{\mathcal{S}}|_V)}$. By Corollary 4.8 we have $\mathcal{S} = \mathcal{T}$. □

Let $X = U \cup V$ with U, V open, quasi-compact subsets. Then $X^* = U^* \cup V^*$ and both U^* and V^* are closed subsets of X^*. Let $Y \in L_{\text{open}}(X^*)$ then $Y = \cup_I Y_i$ with each $D_i := X \setminus Y_i$ open, quasi-compact subset in X. Since each $D_i \cap U$ is an open and quasi-compact subset in U, it follows that $Y \cap U = \cup_I (Y_i \cap U) \in L_{\text{open}}(U^*)$. Then the map

$$\beta_{X,U} : L_{\text{open}}(X^*) \to L_{\text{open}}(U^*), \quad Y \mapsto Y \cap U$$

is a lattice map. The lattice map $\beta_{X,V} : L_{\text{open}}(X^*) \to L_{\text{open}}(V^*)$ is similarly defined.

Lemma 5.2. *The square*

$$\begin{tikzcd} L_{\text{open}}(X^*) \arrow[r, "\beta_{X,U}"] \arrow[d, "\beta_{X,V}"'] & L_{\text{open}}(U^*) \arrow[d, "\beta_{U,U\cap V}"] \\ L_{\text{open}}(V^*) \arrow[r, "\beta_{V,U\cap V}"] & L_{\text{open}}((U\cap V)^*) \end{tikzcd}$$

is commutative and pullback.

Proof. It is easy to see that the lattice maps

$$Y \in L_{\text{open}}(X^*) \mapsto (Y \cap U, Y \cap V) \in L_{\text{open}}(U^*) \prod_{L_{\text{open}}((U\cap V)^*)} L_{\text{open}}(V^*)$$

and

$$(Y_1, Y_2) \in L_{\text{open}}(U^*) \prod_{L_{\text{open}}((U\cap V)^*)} L_{\text{open}}(V^*) \mapsto Y_1 \cup Y_2 \in L_{\text{open}}(X^*)$$

are mutual inverses. □

Lemma 5.3. *Given a subcategory $\mathcal{X}$ in* $\mathrm{Qcoh}(X)$, *we have*

$$\bigcup_{\mathcal{F} \in \mathcal{X}} \mathsf{supp}_X(\mathcal{F}) = \bigcup_{\mathcal{F} \in \sqrt{\mathcal{X}}} \mathsf{supp}_X(\mathcal{F}).$$

Proof. Since $\mathsf{supp}_X(\oplus_I \mathcal{F}_i) = \cup_I \mathsf{supp}_X(\mathcal{F}_i)$ and $\mathsf{supp}_X(\mathcal{F}) = \mathsf{supp}_X(\mathcal{F}') \cup \mathsf{supp}_X(\mathcal{F}'')$ for any short exact sequence $\mathcal{F}' \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{F}''$ in $\mathrm{Qcoh}(X)$, we may assume that $\mathcal{X}$ is closed under subobjects, quotient objects, and direct sums, i.e. $\mathcal{X} = \langle \mathcal{X}^{\oplus} \rangle$. If $\mathcal{F} = \sum_I F_i$ we also have $\mathsf{supp}_X(\mathcal{F}) \subseteq \cup_I \mathsf{supp}_X(\mathcal{F}_i)$. Now our assertion follows from Proposition 2.2. □

Lemma 5.4. *Given a tensor localizing subcategory of finite type $\mathcal{S} \in L_{\text{f.loc},\otimes}(X)$, the set*

$$Y = \bigcup_{\mathcal{F} \in \mathcal{S}} \mathsf{supp}_X(\mathcal{F})$$

is open in X^*.

Proof. We use the Reduction Principle to show that $Y \in L_{\text{open}}(X^*)$. It is the case for affine schemes (see [10, 2.2]). Suppose $X = U \cup V$, where U, V are quasi-compact open subsets of X, and the assertion is true for $U, V, U \cap V$. We have to show that it is true for X itself.

By Corollary 4.8 $\alpha_{X,U}(\mathcal{S}) = \sqrt{(\widehat{\mathcal{S}}|_U)} \in L_{\text{f.loc},\otimes}(U)$ and $\alpha_{X,V}(\mathcal{S}) = \sqrt{(\widehat{\mathcal{S}}|_V)} \in L_{\text{f.loc},\otimes}(V)$. By assumption,

$$Y_1 = \bigcup_{\mathcal{F} \in \sqrt{\widehat{\mathcal{S}}|_U}} \mathsf{supp}_U(\mathcal{F}) \in L_{\text{open}}(U^*)$$

and

$$Y_2 = \bigcup_{\mathcal{F} \in \sqrt{\widehat{\mathcal{S}}|_V}} \mathsf{supp}_V(\mathcal{F}) \in L_{\text{open}}(V^*).$$

By Lemma 5.3

$$Y_1 = \bigcup_{\mathcal{F} \in \widehat{\mathcal{S}}|_U} \mathsf{supp}_U(\mathcal{F}) = \bigcup_{\mathcal{F} \in \mathcal{S}} \mathsf{supp}_U(\mathcal{F}|_U)$$

and

$$Y_2 = \bigcup_{\mathcal{F} \in \widehat{\mathcal{S}}|_V} \mathsf{supp}_V(\mathcal{F}) = \bigcup_{\mathcal{F} \in \mathcal{S}} \mathsf{supp}_V(\mathcal{F}|_V).$$

For every $\mathcal{F} \in \mathrm{Qcoh}(X)$ we have $\mathsf{supp}_X(\mathcal{F}) = \mathsf{supp}_U(\mathcal{F}|_U) \cup \mathsf{supp}_V(\mathcal{F}|_V)$. Therefore $Y_1 = Y \cap U$ and $Y_2 = Y \cap V$. By Lemma 5.2 $Y = Y_1 \cup Y_2 \in L_{\text{open}}(X^*)$. □

We are now in a position to prove the main result of the paper.

Theorem 5.5 (Classification; see Garkusha-Prest [10] for affine schemes)**.** *Let X be a quasi-compact, quasi-separated scheme. Then the maps*

$$Y \overset{\varphi_X}{\longmapsto} \mathcal{S}(Y) = \{\mathcal{F} \in \mathrm{Qcoh}(X) \mid \mathsf{supp}_X(\mathcal{F}) \subseteq Y\}$$

and

$$\mathcal{S} \overset{\psi_X}{\longmapsto} Y(\mathcal{S}) = \bigcup_{\mathcal{F} \in \mathcal{S}} \mathsf{supp}_X(\mathcal{F})$$

induce bijections between

(1) *the set of all subsets of the form $Y = \bigcup_{i \in \Omega} Y_i$ with quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$; that is, the set of all open subsets of X^*,*
(2) *the set of all tensor localizing subcategories of finite type in* $\mathrm{Qcoh}(X)$.

Moreover, $\mathcal{S}(Y) = \sqrt{(\cup_{i \in I} \mathcal{S}(Y_i))} = \sqrt{(\cup_{i \in I} \mathcal{S}_{D_i})}$, where $D_i = X \setminus Y_i$, $\mathcal{S}_{D_i} = \{\mathcal{F} \in$ $\mathrm{Qcoh}(X) \mid \mathcal{F}|_{D_i} = 0\}$.

Proof. By Proposition 5.1 and Lemma 5.4 $\varphi_X(Y) \in L_{\text{f.loc},\otimes}(X)$ and $\psi_X(\mathcal{S}) \in L_{\text{open}}(X^*)$. We have lattice maps

$$\varphi_X : L_{\text{open}}(X^*) \to L_{\text{f.loc},\otimes}(X), \quad \psi_X : L_{\text{f.loc},\otimes}(X) \to L_{\text{open}}(X^*).$$

We use the Reduction Principle to show that $\varphi_X \psi_X = 1$ and $\psi_X \varphi_X = 1$. It is the case for affine schemes (see [10, 2.2]). Suppose $X = U \cup V$, where U, V are quasi-compact open subsets of X, and the assertion is true for $U, V, U \cap V$. We have to show that it is true for X itself.

One has the following commutative diagram of lattices:

$$
\begin{tikzcd}[column sep=small]
& L_{\mathrm{open}}(V^*) \arrow[rr, "\beta_{V,U\cap V}"] \arrow[dd] & & L_{\mathrm{open}}((U\cap V)^*) \arrow[dd, "\varphi_{U\cap V}"] \\
L_{\mathrm{open}}(X^*) \arrow[ur] \arrow[rr] \arrow[dd, "\varphi_X"'] & & L_{\mathrm{open}}(U^*) \arrow[ur] \arrow[dd] & \\
& L_{\mathrm{f.loc},\otimes}(V) \arrow[rr, dotted] \arrow[dd] & & L_{\mathrm{f.loc},\otimes}(U\cap V) \arrow[dd, "\psi_{U\cap V}"] \\
L_{\mathrm{f.loc},\otimes}(X) \arrow[ur] \arrow[rr, "\alpha_{X,U}"] \arrow[dd, "\psi_X"'] & & L_{\mathrm{f.loc},\otimes}(U) \arrow[ur] \arrow[dd] & \\
& L_{\mathrm{open}}(V^*) \arrow[rr, dotted] & & L_{\mathrm{open}}((U\cap V)^*) \\
L_{\mathrm{open}}(X^*) \arrow[ur] \arrow[rr, "\beta_{X,U}"'] & & L_{\mathrm{open}}(U^*) \arrow[ur] &
\end{tikzcd}
$$

By assumption, all vertical arrows except φ_X, ψ_X are bijections. Precisely, the maps φ_U, ψ_U (respectively φ_V, ψ_V and $\varphi_{U\cap V}, \psi_{U\cap V}$) are mutual inverses. Since each horizontal square is pullback (see Corollary 4.8 and Lemma 5.2), it follows that φ_X, ψ_X are mutual inverses.

The fact that $\mathcal{S}(Y) = \sqrt{(\cup_{i\in I}\mathcal{S}(Y_i))} = \sqrt{(\cup_{i\in I}\mathcal{S}|_{D_i})}$ is a consequence of Propositions 2.1 and 5.1. The theorem is proved. $\square$

Denote by $\mathcal{D}_{\mathrm{per}}(X)$ the derived category of perfect complexes, the homotopy category of those complexes of sheaves of $\mathcal{O}_X$-modules which are locally quasi-isomorphic to a bounded complex of free $\mathcal{O}_X$-modules of finite type. We say a thick triangulated subcategory $\mathcal{A} \subset \mathcal{D}_{\mathrm{per}}(X)$ is a *tensor subcategory* if for each object E in $\mathcal{D}_{\mathrm{per}}(X)$ and each A in $\mathcal{A}$, the derived tensor product $E \otimes_X^L A$ is also in $\mathcal{A}$.

Let E be a complex of sheaves of $\mathcal{O}_X$-modules. The *cohomological support* of E is the subspace $\mathsf{supph}_X(E) \subseteq X$ of those points $x \in X$ at which the stalk complex of $\mathcal{O}_{X,x}$-modules E_x is not acyclic. Thus $\mathsf{supph}_X(E) = \bigcup_{n\in\mathbb{Z}} \mathsf{supp}_X(H_n(E))$ is the union of the supports in the classic sense of the cohomology sheaves of E.

We shall write $L_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(X))$ to denote the lattice of all thick subcategories of $\mathcal{D}_{\mathrm{per}}(X)$.

Theorem 5.6 (Thomason [26]). *Let X be a quasi-compact and quasi-separated scheme. The assignments*

$$\mathcal{T} \in L_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(X)) \overset{\mu}{\longmapsto} Y(\mathcal{T}) = \bigcup_{E\in\mathcal{T}} \mathsf{supph}_X(E)$$

and

$$Y \in L_{\mathrm{open}}(X^*) \overset{\nu}{\longmapsto} \mathcal{T}(Y) = \{E \in \mathcal{D}_{\mathrm{per}}(X) \mid \mathsf{supph}_X(E) \subseteq Y\}$$

are mutually inverse lattice isomorphisms.

The next result says that there is a 1-1 correspondence between the tensor thick subcategories of perfect complexes and the tensor localizing subcategories of finite type of quasi-coherent sheaves.

Theorem 5.7 (see Garkusha-Prest [10] for affine schemes). *Let X be a quasi-compact and quasi-separated scheme. The assignments*

$$\mathcal{T} \in L_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(X)) \overset{\rho}{\longmapsto} \mathcal{S} = \{\mathcal{F} \in \mathrm{Qcoh}(X) \mid \mathsf{supp}_X(\mathcal{F}) \subseteq Y(\mathcal{T})\}$$

and

$$\mathcal{S} \in L_{\text{f.loc},\otimes}(X) \stackrel{\tau}{\longmapsto} \{E \in \mathcal{D}_{\text{per}}(X) \mid H_n(E) \in \mathcal{S} \textit{ for all } n \in \mathbb{Z}\}$$

are mutually inverse lattice isomorphisms.

Proof. Consider the following diagram

$$\begin{tikzcd} & L_{\text{open}}(X^*) \arrow[dl, "\varphi"'] \arrow[dr, "\nu"] & \\ L_{\text{f.loc},\otimes}(X) \arrow[rr, shift left, "\tau"] & & L_{\text{thick}}(\mathcal{D}_{\text{per}}(X)), \arrow[ll, shift left, "\rho"] \end{tikzcd}$$

in which φ,ν are the lattice maps described in Theorems 5.5 and 5.6. Using the fact that $\mathsf{supph}_X(E) = \bigcup_{n\in\mathbb{Z}} \mathsf{supp}_X(H_n(E))$, $E \in \mathcal{D}_{\text{per}}(X)$, and Theorems 5.5, 5.6 one sees that $\tau\varphi = \nu$ and $\rho\nu = \varphi$. Then $\rho\tau = \rho\nu\varphi^{-1} = \varphi\varphi^{-1} = 1$ and $\tau\rho = \tau\varphi\nu^{-1} = \nu\nu^{-1} = 1$. $\square$

6. THE ZARISKI TOPOLOGY ON $\mathsf{Sp}(X)$

We are going to construct two maps

$$\alpha : X \to \mathsf{Sp}(X) \quad \text{and} \quad \beta : \mathsf{Sp}(X) \to X.$$

Given $P \in X$ there is an affine neighborhood $U = \mathsf{Spec}\,R$ of P. Let E_P denote the injective hull of the quotient module R/P. Then E_P is an indecomposable injective R-module. By Proposition 2.1 $\operatorname{Mod} R$ can be regarded as the quotient category $\operatorname{Qcoh}(X)/\mathcal{S}_U$, where $\mathcal{S}_U = \operatorname{Ker} j_U^*$ with $j_U : U \to X$ the canonical injection. Therefore $j_{U,*} : \operatorname{Mod} R \to \operatorname{Qcoh}(X)$ takes injectives to injectives. We set $\alpha(P) = j_{U,*}(E_P) \in \mathsf{Sp}(X)$.

The definition of α does not depend on choice of the affine neighborhood U. Indeed, let $P \in V = \mathsf{Spec}\,S$ with S a commutative ring. Then $j_{U,*}(E_P) \cong j_{V,*}(E_P) \cong j_{U\cap V,*}(E_P)$, hence these represent the same element in $\mathsf{Sp}(X)$. We denote it by $\mathcal{E}_P$.

Now let us define the map β. Let $X = \cup_{i=1}^n U_i$ with each $U_i = \mathsf{Spec}\,R_i$ an affine scheme and let $\mathcal{E} \in \mathsf{Sp}(X)$. Then $\mathcal{E}$ has no $\mathcal{S}_{U_i}$-torsion for some $i \le n$, because $\cap_{i=1}^n \mathcal{S}_{U_i} = 0$ and $\mathcal{E}$ is uniform. Since $\operatorname{Mod} R_i$ is equivalent to $\operatorname{Qcoh}(X)/\mathcal{S}_{U_i}$, $\mathcal{E}$ can be regarded as an indecomposable injective R_i-module. Set $P = P(\mathcal{E})$ to be the sum of annihilator ideals in R_i of non-zero elements, equivalently non-zero submodules, of $\mathcal{E}$. Since $\mathcal{E}$ is uniform the set of annihilator ideals of non-zero elements of $\mathcal{E}$ is closed under finite sum. It is easy to check ([23, 9.2]) that $P(\mathcal{E})$ is a prime ideal. By construction, $P(\mathcal{E}) \in U_i$. Clearly, the definition of $P(\mathcal{E})$ does not depend on choice of U_i and $P(\mathcal{E}_P) = P$. We see that $\beta\alpha = 1_X$. In particular, α is an embedding of X into $\mathsf{Sp}(X)$. We shall consider this embedding as identification.

Given a commutative coherent ring R and an indecomposable injective R-module $E \in \mathsf{Sp}\,R$, Prest [23, 9.6] observed that E is elementary equivalent to $E_{P(E)}$ in the first order language of modules. Translating this fact from model-theoretic idioms to algebraic language, it says that every localizing subcategory of finite type $\mathcal{S} \in L_{\text{f.loc}}(\operatorname{Mod} R)$ is cogenerated by prime ideals. More precisely, there is a set $D \subset \mathsf{Spec}\,R$ such that $S \in \mathcal{S}$ if and only if $\operatorname{Hom}_R(S, E_P) = 0$ for all $P \in D$. This has been generalized to all commutative rings by Garkusha-Prest [10]. Moreover, $D = \mathsf{Spec}\,R \setminus \bigcup_{S\in\mathcal{S}} \mathsf{supp}_R(S)$.

Proposition 6.1. *Let $\mathcal{E} \in \mathsf{Sp}(X)$ and let $P(\mathcal{E}) \in X$ be the point defined above. Then $\mathcal{E}$ and $\mathcal{E}_{P(\mathcal{E})}$ are topologically indistinguishable in $\mathsf{Sp}_{fl}(X)$. In other words, for every $\mathcal{S} \in L_{\text{f.loc}}(X)$ the sheaf $\mathcal{E}$ has no $\mathcal{S}$-torsion if and only if $\mathcal{E}_{P(\mathcal{E})}$ has no $\mathcal{S}$-torsion.*

Proof. Let $U = \mathsf{Spec}\, R \subset X$ be such that $\mathcal{E}$ has no $\mathcal{S}_U$-torsion. Then $P(\mathcal{E}) \in U$ and $\mathcal{E}$ and $\mathcal{E}_{P(\mathcal{E})}$ have no $\mathcal{S}_U$-torsion. These can also be considered as indecomposable injective R-modules, because $\mathrm{Qcoh}(X)/\mathcal{S}_U \cong \mathrm{Mod}\, R$ by Proposition 2.1. Denote by $\mathcal{S}' = \alpha_{X,U}(\mathcal{S}) \in L_{\text{f.loc}}(\mathrm{Mod}\, R)$. Then $\mathcal{E}$ has no $\mathcal{S}$-torsion in $\mathrm{Qcoh}(X)$ if and only if $\mathcal{E}$ has no $\mathcal{S}'$-torsion in $\mathrm{Mod}\, R$. Our assertion now follows from [10, 3.5]. $\square$

Corollary 6.2. *If $\mathcal{S} \in L_{\text{f.loc},\otimes}(X)$ then $\mathcal{O}(\mathcal{S}) \bigcap X = Y(\mathcal{S})$, where $Y(\mathcal{S}) = \bigcup_{\mathcal{F} \in \mathcal{S}} \mathsf{supp}_X(\mathcal{F}) \in L_{\text{open}}(X^*)$.*

Proof. If $\mathcal{E} \in \mathcal{O}(\mathcal{S})$ and $U = \mathsf{Spec}\, R \subset X$ is such that $\mathcal{E}$ has no $\mathcal{S}_U$-torsion, then $P(\mathcal{E}) \in U$ and $\mathcal{E}_{P(\mathcal{E})} \in \mathcal{O}(\mathcal{S})$ by Proposition 6.1. Let $\mathcal{S}' = \alpha_{X,U}(\mathcal{S}) \in L_{\text{f.loc}}(\mathrm{Mod}\, R)$. We have $Y(\mathcal{S}') := \bigcup_{S \in \mathcal{S}} \mathsf{supp}_R(S) \subset Y(\mathcal{S})$. By the proof of Proposition 6.1 $\mathcal{E}_{P(\mathcal{E})}$ has $\mathcal{S}'$-torsion. Then there is a finitely generated ideal $I \subset R$ such that $R/I \in \mathcal{S}'$ and $\mathrm{Hom}_R(R/I, \mathcal{E}_{P(\mathcal{E})}) \neq 0$. It follows from [10, 3.4] that $P(\mathcal{E}) \in Y(\mathcal{S}')$, and hence $\mathcal{O}(\mathcal{S}) \bigcap X \subset Y(\mathcal{S})$.

Conversely, if $P \in Y(\mathcal{S}) \cap U$ then $\mathcal{E}_P$ has $\mathcal{S}'$-torsion by [10, 3.4]. Therefore $\mathcal{E}_P \in \mathcal{O}(\mathcal{S}') \subset \mathcal{O}(\mathcal{S})$. It immediately follows that $\mathcal{O}(\mathcal{S}) \bigcap X \supset Y(\mathcal{S})$. $\square$

Proposition 6.3. *(cf.* [10, 3.7]*) Let X be a quasi-compact and quasi-separated scheme. Then the maps*

$$Y \in L_{\text{open}}(X^*) \overset{\sigma}{\mapsto} \mathcal{O}_Y = \{\mathcal{E} \in \mathsf{Sp}(X) \mid P(\mathcal{E}) \in Y\}$$

and

$$\mathcal{O} \in L_{\text{open}}(\mathsf{Sp}_{fl,\otimes}(X)) \overset{\varepsilon}{\mapsto} Y_{\mathcal{O}} = \{P(\mathcal{E}) \in X^* \mid \mathcal{E} \in \mathcal{O}\} = \mathcal{O} \cap X^*$$

induce a 1-1 correspondence between the lattices of open sets of X^* and those of $\mathsf{Sp}_{fl,\otimes}(X)$.

Proof. Let $\mathcal{S}(Y) = \{\mathcal{F} \in \mathrm{Qcoh}(X) \mid \mathsf{supp}_X(\mathcal{F}) \subseteq Y\} \in L_{\text{f.loc},\otimes}(X)$; then $Y = Y(\mathcal{S}(Y))$ by the Classification Theorem and $\mathcal{O}(\mathcal{S}(Y)) \bigcap X = Y$ by Corollary 6.2. It follows that $\mathcal{O}(\mathcal{S}(Y)) \subseteq \mathcal{O}_Y$. On the other hand, if $\mathcal{E} \in \mathcal{O}_Y$ then the proof of Corollary 6.2 shows that $\mathcal{E}_{P(\mathcal{E})} \in \mathcal{O}(\mathcal{S}(Y))$. Proposition 6.1 implies $\mathcal{E} \in \mathcal{O}(\mathcal{S}(Y))$, hence $\mathcal{O}(\mathcal{S}(Y)) \supseteq \mathcal{O}_Y$. We see that $\mathcal{O}_Y = \mathcal{O}(\mathcal{S}(Y)) \in L_{\text{open}}(\mathsf{Sp}_{fl,\otimes}(X))$.

Let $\mathcal{O} \in L_{\text{open}}(\mathsf{Sp}_{fl,\otimes}(X))$. By Theorem 4.7 there is a unique $\mathcal{S} \in L_{\text{f.loc},\otimes}(X)$ such that $\mathcal{O} = \mathcal{O}(\mathcal{S})$. By Corollary 6.2 $\mathcal{O} \bigcap X = Y(\mathcal{S}) = Y_{\mathcal{O}}$, and so $Y_{\mathcal{O}} \in L_{\text{open}}(X^*)$. It is now easy to verify that $Y_{\mathcal{O}_Y} = Y$ and $\mathcal{O}_{Y_{\mathcal{O}}} = \mathcal{O}$. $\square$

We notice that a subset $Y \subset X^*$ is open and quasi-compact in X^* if and only if $X \setminus Y$ is an open and quasi-compact subset in X.

Proposition 6.4. *An open subset $\mathcal{O} \in L_{\text{open}}(\mathsf{Sp}_{fl,\otimes}(X))$ is quasi-compact if and only if it is of the form $\mathcal{O} = \mathcal{O}(\mathcal{S}(Y))$ with Y an open and quasi-compact subset in X^*. The space $\mathsf{Sp}_{fl,\otimes}(X)$ is quasi-compact, the quasi-compact open subsets are closed under finite intersections and form an open basis, and every non-empty irreducible closed subset has a generic point.*

Proof. Let $\mathcal{O} \in L_{\text{open}}(\mathsf{Sp}_{fl,\otimes}(X))$ be quasi-compact. By Theorem 4.7 there is a unique $\mathcal{S} \in L_{\text{f.loc},\otimes}(X)$ such that $\mathcal{O} = \mathcal{O}(\mathcal{S})$. By the Classification Theorem $\mathcal{S} =$

$\mathcal{S}(Y)=\sqrt{}(\cup_I\mathcal{S}(Y_i))$, where $Y=\bigcup_{\mathcal{F}\in\mathcal{S}}\mathsf{supp}_X(\mathcal{F})\in L_{\text{open}}(X^*)$, each Y_i is such that $X\setminus Y_i$ is open and quasi-compact subset of X, and $Y=\bigcup_I Y_i$. Then $\mathcal{O}=\bigcup_I\mathcal{O}(\mathcal{S}(Y_i))$. Since $\mathcal{O}$ is quasi-compact, there is a finite subset $J\subset I$ such that $\mathcal{O}=\bigcup_J\mathcal{O}(\mathcal{S}(Y_i))=\mathcal{O}(\sqrt{}(\cup_J\mathcal{S}(Y_i)))=\mathcal{O}(\mathcal{S}(\cup_J Y_i))$. Since X is spectral, then $X\setminus(\cup_J Y_i)=\cap_J(X\setminus Y_i)$ is an open and quasi-compact subset in X.

Conversely, let $\mathcal{O}=\mathcal{O}(\mathcal{S}(Y))$ with $X\setminus Y$ an open and quasi-compact subset in X and let $\mathcal{O}=\bigcup_I\mathcal{O}_i$ with each $\mathcal{O}_i\in L_{\text{open}}(\mathsf{Sp}_{fl,\otimes}(X))$. By Theorem 4.7 there are unique $\mathcal{S}_i\in L_{\text{f.loc},\otimes}(X)$ such that $\mathcal{O}_i=\mathcal{O}(\mathcal{S}_i)$ and $\mathcal{S}(Y)=\sqrt{}(\cup_I\mathcal{S}_i)$. We set $Y_i=\bigcup_{\mathcal{F}\in\mathcal{S}_i}\mathsf{supp}_X(\mathcal{F})$ for each $i\in I$. By Lemma 5.3 and the Classification Theorem one has $Y=\bigcup_I Y_i$. Since Y is quasi-compact in X^*, there is a finite subset $J\subset I$ such that $Y=\bigcup_J Y_i$. It follows that $\mathcal{S}(Y)=\sqrt{}(\cup_J\mathcal{S}_i)$ and $\mathcal{O}=\bigcup_J\mathcal{O}_i$.

The space $\mathsf{Sp}_{fl,\otimes}(X)$ is quasi-compact, because it equals $\mathcal{O}(\mathcal{S}(X^*))$ and X^* is quasi-compact. The quasi-compact open subsets are closed under finite intersections, because $\mathcal{O}(\mathcal{S}(Y_1))\cap\mathcal{O}(\mathcal{S}(Y_2))=\mathcal{O}(\mathcal{S}(Y_1\cap Y_2))$ with Y_1,Y_2 open and quasi-compact subsets in X^*. Since $\mathcal{O}(\mathcal{S}(Y))=\cup_I\mathcal{O}(\mathcal{S}(Y_i))$, where $Y=\cup_I Y_i$ and each Y_i is an open and quasi-compact subset in X^*, the quasi-compact open subsets also form an open basis.

Finally, it follows from Corollary 6.2 that a subset U of $\mathsf{Sp}_{fl,\otimes}(X)$ is closed and irreducible if and only if so is $\widehat{U}:=U\cap X^*$. Since X^* is spectral then $\widehat{U}$ has a generic point P. The point $\mathcal{E}_P\in U$ is generic. $\square$

Though the space $\mathsf{Sp}_{fl,\otimes}(X)$ is not in general T_0 (see [8]), nevertheless we make the same definition for $(\mathsf{Sp}_{fl,\otimes}(X))^*$ as for spectral spaces and denote it by $\mathsf{Sp}_{zar}(X)$. By definition, $\mathcal{Q}\in L_{\text{open}}(\mathsf{Sp}_{zar}(X))$ if and only if $\mathcal{Q}=\cup_I\mathcal{Q}_i$ with each $\mathcal{Q}_i$ having quasi-compact and open complement in $\mathsf{Sp}_{fl,\otimes}(X)$. The topology on $\mathsf{Sp}_{zar}(X)$ will also be referred to as the *Zariski topology*. Notice that the Zariski topology on $\mathsf{Sp}_{zar}(\mathsf{Spec}\,R)$, R is coherent, concides with the Zariski topology on the injective spectrum $\mathsf{Sp}\,R$ in the sense of Prest [23].

Theorem 6.5 (cf. Garkusha-Prest [8, 9, 10])**.** *Let X be a quasi-compact and quasi-separated scheme. The space X is dense and a retract in $\mathsf{Sp}_{zar}(X)$. A left inverse to the embedding $X\hookrightarrow\mathsf{Sp}_{zar}(X)$ takes $\mathcal{E}\in\mathsf{Sp}_{zar}(X)$ to $P(\mathcal{E})\in X$. Moreover, $\mathsf{Sp}_{zar}(X)$ is quasi-compact, the basic open subsets $\mathcal{Q}$, with $\mathsf{Sp}(X)\setminus\mathcal{Q}$ quasi-compact and open subset in $\mathsf{Sp}_{fl,\otimes}(X)$, are quasi-compact, the intersection of two quasi-compact open subsets is quasi-compact, and every non-empty irreducible closed subset has a generic point.*

Proof. Let $\mathcal{Q}\in L_{\text{open}}(\mathsf{Sp}_{zar}(X))$ be such that $\mathcal{O}:=\mathsf{Sp}(X)\setminus\mathcal{Q}$ is a quasi-compact and open subset in $\mathsf{Sp}_{fl,\otimes}(X)$ and let $Y=\mathcal{O}\cap X$ and $D=X\setminus Y=\mathcal{Q}\cap X$. Since Y is a quasi-compact subset in X^*, then D is a quasi-compact subset in X. Notice that $\mathcal{O}=\mathcal{O}(\mathcal{S}_D)$, where $\mathcal{S}_D=\{\mathcal{F}\in\operatorname{Qcoh}(X)\mid\mathcal{F}|_D=0\}$. Clearly, X is dense in $\mathsf{Sp}_{zar}(X)$ and $\alpha:X\to\mathsf{Sp}_{zar}(X)$ is a continuous map.

The map $\beta:\mathsf{Sp}_{zar}(X)\to X$, $\mathcal{E}\mapsto P(\mathcal{E})$, is left inverse to α. Obviously, β is continuous. Thus X is a retract of $\mathsf{Sp}_{zar}(X)$.

Let us show that the basic open set $\mathcal{Q}$ is quasi-compact. Let $\mathcal{Q}=\bigcup_{i\in\Omega}\mathcal{Q}_i$ with each $\mathsf{Sp}(X)\setminus\mathcal{Q}_i$ a quasi-compact and open subset in $\mathsf{Sp}_{fl,\otimes}(X)$ and $D_i:=\mathcal{Q}_i\cap X$. Since D is quasi-compact, then $D=\bigcup_{i\in\Omega_0}D_i$ for some finite subset $\Omega_0\subset\Omega$.

Assume $\mathcal{E} \in \mathcal{Q} \setminus \bigcup_{i\in\Omega_0} \mathcal{Q}_i$. It follows from Proposition 6.3 that $P(\mathcal{E}) \in \mathcal{Q} \cap X = D = \bigcup_{i\in\Omega_0} D_i$. Proposition 6.3 implies that $\mathcal{E} \in \mathcal{Q}_{i_0}$ for some $i_0 \in \Omega_0$, a contradiction. So $\mathcal{Q}$ is quasi-compact. It also follows that the intersection of two quasi-compact open subsets is quasi-compact and that $\mathsf{Sp}_{zar}(X)$ is quasi-compact.

Finally, it follows from Corollary 6.2 that a subset U of $\mathsf{Sp}_{zar}(X)$ is closed and irreducible if and only if so is $\widehat{U} := U \cap X$. Since X is spectral then $\widehat{U}$ has a generic point P. The point $\mathcal{E}_P \in U$ is generic. $\square$

Corollary 6.6. *Let X be a quasi-compact and quasi-separated scheme. The following relations hold:*

$$\mathsf{Sp}_{zar}(X) = (\mathsf{Sp}_{fl,\otimes}(X))^* \textit{ and } \mathsf{Sp}_{fl,\otimes}(X) = (\mathsf{Sp}_{zar}(X))^*.$$

Though the space $\mathsf{Sp}_{zar}(X)$ is strictly bigger than X in general (see [8]), their lattices of open subsets are isomorphic. More precisely, Proposition 6.3 implies that the maps

$$D \in L_{\text{open}}(X) \mapsto \mathcal{Q}_D = \{\mathcal{E} \in \mathsf{Sp}(X) \mid P(\mathcal{E}) \in D\}$$

and

$$\mathcal{Q} \in L_{\text{open}}(\mathsf{Sp}_{zar}(X)) \mapsto D_{\mathcal{Q}} = \{P(\mathcal{E}) \in X \mid \mathcal{E} \in \mathcal{Q}\} = \mathcal{Q} \cap X$$

induce a 1-1 correspondence between the lattices of open sets of X and those of $\mathsf{Sp}_{zar}(X)$. Moreover, sheaves do not see any difference between X and $\mathsf{Sp}_{zar}(X)$. Namely, the following is true.

Proposition 6.7. *Let X be a quasi-compact and quasi-separated scheme. Then the maps of topological spaces $\alpha : X \to \mathsf{Sp}_{zar}(X)$ and $\beta : \mathsf{Sp}_{zar}(X) \to X$ induce isomorphisms of the categories of sheaves*

$$\beta_* : Sh(\mathsf{Sp}_{zar}(X)) \xrightarrow{\cong} Sh(X), \quad \alpha_* : Sh(X) \xrightarrow{\cong} Sh(\mathsf{Sp}_{zar}(X)).$$

Proof. Since $\beta\alpha = 1$ it follows that $\beta_*\alpha_* = 1$. By definition, $\beta_*(\mathcal{F})(D) = \mathcal{F}(\mathcal{Q}_D)$ for any $\mathcal{F} \in Sh(\mathsf{Sp}_{zar}(X)), D \in L_{\text{open}}(X)$ and $\alpha_*(\mathcal{G})(\mathcal{Q}) = \mathcal{G}(D_{\mathcal{Q}})$ for any $\mathcal{G} \in Sh(X), \mathcal{Q} \in L_{\text{open}}(\mathsf{Sp}_{zar}(X))$. We have:

$$\alpha_*\beta_*(\mathcal{F})(\mathcal{Q}) = \beta_*(\mathcal{F})(D_{\mathcal{Q}}) = \mathcal{F}(\mathcal{Q}_{D_{\mathcal{Q}}}) = \mathcal{F}(\mathcal{Q}).$$

We see that $\alpha_*\beta_* = 1$, and so α_*, β_* are mutually inverse isomorphisms. $\square$

Let $\mathcal{O}_{\mathsf{Sp}_{zar}(X)}$ denote the sheaf of commutative rings $\alpha_*(\mathcal{O}_X)$; then $(\mathsf{Sp}_{zar}(X), \mathcal{O}_{\mathsf{Sp}_{zar}(X)})$ is plainly a locally ringed space. If we set $\alpha^\sharp : \mathcal{O}_{\mathsf{Sp}_{zar}(X)} \to \alpha_*\mathcal{O}_X$ and $\beta^\sharp : \mathcal{O}_X \to \beta_*\mathcal{O}_{\mathsf{Sp}_{zar}(X)}$ to be the identity maps, then the map of locally ringed spaces

$$(\alpha, \alpha^\sharp) : (X, \mathcal{O}_X) \to (\mathsf{Sp}_{zar}(X), \mathcal{O}_{\mathsf{Sp}_{zar}(X)})$$

is right inverse to

$$(\beta, \beta^\sharp) : (\mathsf{Sp}_{zar}(X), \mathcal{O}_{\mathsf{Sp}_{zar}(X)}) \to (X, \mathcal{O}_X).$$

Observe that it is *not* a scheme in general, because $\mathsf{Sp}_{zar}(X)$ is not a T_0-space. Proposition 6.7 implies that the categories of the $\mathcal{O}_{\mathsf{Sp}_{zar}(X)}$-modules and $\mathcal{O}_X$-modules are naturally isomorphic.

7. The prime spectrum of an ideal lattice

Inspired by recent work of Balmer [2], Buan, Krause, and Solberg [4] introduce the notion of an ideal lattice and study its prime ideal spectrum. Applications arise from abelian or triangulated tensor categories.

Definition (Buan, Krause, Solberg [4])**.** An *ideal lattice* is by definition a partially ordered set $L = (L, \leqslant)$, together with an associative multiplication $L \times L \to L$, such that the following holds.

(L1) The poset L is a *complete lattice*, that is,

$$\sup A = \bigvee_{a \in A} a \quad \text{and} \quad \inf A = \bigwedge_{a \in A} a$$

exist in L for every subset $A \subseteq L$.

(L2) The lattice L is *compactly generated*, that is, every element in L is the supremum of a set of compact elements. (An element $a \in L$ is *compact*, if for all $A \subseteq L$ with $a \leqslant \sup A$ there exists some finite $A' \subseteq A$ with $a \leqslant \sup A'$.)

(L3) We have for all $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The element $1 = \sup L$ is compact, and $1a = a = a1$ for all $a \in L$.

(L5) The product of two compact elements is again compact.

A *morphism* $\varphi\colon L \to L'$ of ideal lattices is a map satisfying

$$\varphi(\bigvee_{a \in A} a) = \bigvee_{a \in A} \varphi(a) \quad \text{for} \quad A \subseteq L,$$
$$\varphi(1) = 1 \quad \text{and} \quad \varphi(ab) = \varphi(a)\varphi(b) \quad \text{for} \quad a, b \in L.$$

Let L be an ideal lattice. Following [4] we define the spectrum of prime elements in L. An element $p \neq 1$ in L is *prime* if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ for all $a, b \in L$. We denote by $\operatorname{Spec} L$ the set of prime elements in L and define for each $a \in L$

$$V(a) = \{p \in \operatorname{Spec} L \mid a \leqslant p\} \quad \text{and} \quad D(a) = \{p \in \operatorname{Spec} L \mid a \not\leqslant p\}.$$

The subsets of $\operatorname{Spec} L$ of the form $V(a)$ are closed under forming arbitrary intersections and finite unions. More precisely,

$$V(\bigvee_{i \in \Omega} a_i) = \bigcap_{i \in \Omega} V(a_i) \quad \text{and} \quad V(ab) = V(a) \cup V(b).$$

Thus we obtain the *Zariski topology* on $\operatorname{Spec} L$ by declaring a subset of $\operatorname{Spec} L$ to be *closed* if it is of the form $V(a)$ for some $a \in L$. The set $\operatorname{Spec} L$ endowed with this topology is called the *prime spectrum* of L. Note that the sets of the form $D(a)$ with compact $a \in L$ form a basis of open sets. The prime spectrum $\operatorname{Spec} L$ of an ideal lattice L is spectral [4, 2.5].

There is a close relation between spectral spaces and ideal lattices. Given a topological space X, we denote by $L_{\mathrm{open}}(X)$ the lattice of open subsets of X and consider the multiplication map

$$L_{\mathrm{open}}(X) \times L_{\mathrm{open}}(X) \to L_{\mathrm{open}}(X), \quad (U, V) \mapsto UV = U \cap V.$$

The lattice $L_{\mathrm{open}}(X)$ is complete.

The following result, which appears in [4], is part of the Stone Duality Theorem (see, for instance, [17]).

Proposition 7.1. *Let X be a spectral space. Then $L_{\text{open}}(X)$ is an ideal lattice. Moreover, the map*

$$X \to \operatorname{Spec} L_{\text{open}}(X), \quad x \mapsto X \setminus \overline{\{x\}},$$

is a homeomorphism.

We deduce from the Classification Theorem the following.

Proposition 7.2. *Let X be a quasi-compact and quasi-separated scheme. Then $L_{\text{f.loc},\otimes}(X)$ is an ideal lattice.*

Proof. The space X is spectral. Thus X^* is spectral, also $L_{\text{open}}(X^*)$ is an ideal lattice by Proposition 7.1. By the Classification Theorem we have an isomorphism $L_{\text{open}}(X^*) \cong L_{\text{f.loc},\otimes}(X)$. Therefore $L_{\text{f.loc},\otimes}(X)$ is an ideal lattice. □

It follows from Proposition 6.4 that $\mathcal{S} \in L_{\text{f.loc},\otimes}(X)$ is compact if and only if $\mathcal{S} = \mathcal{S}(Y)$ with $Y \in L_{\text{open}}(X^*)$ compact.

Corollary 7.3. *Let X be a quasi-compact and quasi-separated scheme. The points of $\operatorname{Spec} L_{\text{f.loc},\otimes}(X)$ are the $\cap$-irreducible tensor localizing subcategories of finite type in $\operatorname{Qcoh}(X)$ and the map*

$$f : X^* \to \operatorname{Spec} L_{\text{f.loc},\otimes}(X), \quad P \mapsto \mathcal{S}_P = \{\mathcal{F} \in \operatorname{Qcoh}(X) \mid \mathcal{F}_P = 0\}$$

is a homeomorphism of spaces.

Proof. This is a consequence of the Classification Theorem and Propositions 7.1, 7.2. □

8. Reconstructing quasi-compact, quasi-separated schemes

Let X be a quasi-compact and quasi-separated scheme. We shall write $\operatorname{Spec}(\operatorname{Qcoh}(X)) := (\operatorname{Spec} L_{\text{f.loc},\otimes}(X))^*$ and $\operatorname{supp}(\mathcal{F}) := \{\mathcal{P} \in \operatorname{Spec}(\operatorname{Qcoh}(X)) \mid \mathcal{F} \notin \mathcal{P}\}$ for $\mathcal{F} \in \operatorname{Qcoh}(X)$. It follows from Corollary 7.3 that

$$\operatorname{supp}_X(\mathcal{F}) = f^{-1}(\operatorname{supp}(\mathcal{F})).$$

Following [2, 4], we define a structure sheaf on $\operatorname{Spec}(\operatorname{Qcoh}(X))$ as follows. For an open subset $U \subseteq \operatorname{Spec}(\operatorname{Qcoh}(X))$, let

$$\mathcal{S}_U = \{\mathcal{F} \in \operatorname{Qcoh}(X) \mid \operatorname{supp}(\mathcal{F}) \cap U = \emptyset\}$$

and observe that $\mathcal{S}_U = \{\mathcal{F} \mid \mathcal{F}_P = 0 \text{ for all } P \in f^{-1}(U)\}$ is a tensor localizing subcategory. We obtain a presheaf of rings on $\operatorname{Spec}(\operatorname{Qcoh}(X))$ by

$$U \mapsto \operatorname{End}_{\operatorname{Qcoh}(X)/\mathcal{S}_U}(\mathcal{O}_X).$$

If $V \subseteq U$ are open subsets, then the restriction map

$$\operatorname{End}_{\operatorname{Qcoh}(X)/\mathcal{S}_U}(\mathcal{O}_X) \to \operatorname{End}_{\operatorname{Qcoh}(X)/\mathcal{S}_V}(\mathcal{O}_X)$$

is induced by the quotient functor $\operatorname{Qcoh}(X)/\mathcal{S}_U \to \operatorname{Qcoh}(X)/\mathcal{S}_V$. The sheafification is called the *structure sheaf* of $\operatorname{Qcoh}(X)$ and is denoted by $\mathcal{O}_{\operatorname{Qcoh}(X)}$. Next let $\mathcal{P} \in \operatorname{Spec}(\operatorname{Qcoh}(X))$ and $P := f^{-1}(\mathcal{P})$. There is an affine neighborhood $\operatorname{Spec} R$ of P. We have

$$\mathcal{O}_{\operatorname{Qcoh}(X),\mathcal{P}} \cong \varinjlim_{\mathcal{P} \in V} \operatorname{End}_{\operatorname{Mod} R/\mathcal{S}_V}(R) \cong R_P \cong \mathcal{O}_{X,P}.$$

The second isomorphism follows from [10, §8]. We see that each stalk $\mathcal{O}_{\operatorname{Qcoh}(X),\mathcal{P}}$ is a commutative ring. We claim that $\mathcal{O}_{\operatorname{Qcoh}(X)}$ is a sheaf of commutative rings.

Indeed, let $a, b \in \mathcal{O}_{\mathrm{Qcoh}(X)}(U)$, where $U \in L_{\mathrm{open}}(\mathsf{Spec}(\mathrm{Qcoh}(X)))$. For all $\mathcal{P} \in U$ we have $\rho^U_{\mathcal{P}}(ab) = \rho^U_{\mathcal{P}}(ba)$, where $\rho^U_{\mathcal{P}} : \mathcal{O}_{\mathrm{Qcoh}(X)}(U) \to \mathcal{O}_{\mathrm{Qcoh}(X),\mathcal{P}}$ is the natural homomorphism. Since $\mathcal{O}_{\mathrm{Qcoh}(X)}$ is a sheaf, it follows that $ab = ba$.

The next theorem says that the abelian category $\mathrm{Qcoh}(X)$ contains all the necessary information to reconstruct the scheme $(X, \mathcal{O}_X)$.

Theorem 8.1 (Reconstruction; cf. Rosenberg [24])**.** *Let X be a quasi-compact and quasi-separated scheme. The map of Corollary 7.3 induces an isomorphism of ringed spaces*

$$f : (X, \mathcal{O}_X) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{Qcoh}(X)), \mathcal{O}_{\mathrm{Qcoh}(X)}).$$

Proof. The proof is similar to that of [4, 8.3; 9.4]. Fix an open subset $U \subseteq \mathsf{Spec}(\mathrm{Qcoh}(X))$ and consider the functor

$$F : \mathrm{Qcoh}(X) \xrightarrow{(-)|_{f^{-1}(U)}} \mathrm{Qcoh}\, f^{-1}(U).$$

We claim that F annihilates $\mathcal{S}_U$. In fact, $\mathcal{F} \in \mathcal{S}_U$ implies $f^{-1}(\mathsf{supp}(\mathcal{F})) \cap f^{-1}(U) = \emptyset$ and therefore $\mathsf{supp}_X(\mathcal{F}) \cap f^{-1}(U) = \emptyset$. Thus $\mathcal{F}_P = 0$ for all $P \in f^{-1}(U)$ and therefore $F(\mathcal{F}) = 0$. It follows that F factors through $\mathrm{Qcoh}(X)/\mathcal{S}_U$ and induces a map $\mathrm{End}_{\mathrm{Qcoh}(X)/\mathcal{S}_U}(\mathcal{O}_X) \to \mathcal{O}_X(f^{-1}(U))$ which extends to a map $\mathcal{O}_{\mathrm{Qcoh}(X)}(U) \to \mathcal{O}_X(f^{-1}(U))$. This yields the morphism of sheaves $f^\sharp \colon \mathcal{O}_{\mathrm{Qcoh}(X)} \to f_* \mathcal{O}_X$.

By the above $f^\sharp$ induces an isomorphism $f^\sharp_P \colon \mathcal{O}_{\mathrm{Qcoh}(X), f(P)} \to \mathcal{O}_{X,P}$ at each point $P \in X$. We conclude that $f^\sharp_P$ is an isomorphism. It follows that f is an isomorphism of ringed spaces if the map $f : X \to \mathsf{Spec}(\mathrm{Qcoh}(X))$ is a homeomorphism. This last condition is a consequence of Propositions 7.1-7.2 and Corollary 7.3. □

9. Coherent schemes

We end up the paper with introducing coherent schemes. These are between noetherian and quasi-compact, quasi-separated schemes and generalize commutative coherent rings. We want to obtain the Classification and Reconstruction results for such schemes.

Definition. A scheme X is *locally coherent* if it can be covered by open affine subsets $\mathsf{Spec}\, R_i$, where each R_i is a coherent ring. X is *coherent* if it is locally coherent, quasi-compact and quasi-separated.

The trivial example of a coherent scheme is $\mathsf{Spec}\, R$ with R a coherent ring. There is a plenty of coherent rings. For instance, let R be a noetherian ring, and X be any (possibly infinite) set of commuting indeterminates. Then the polynomial ring $R[X]$ is coherent. As a note of caution, however, we should point out that, in general, the coherence of a ring R does not imply that of $R[x]$ for one variable x. In fact, if R is a countable product of the polynomial ring $\mathbb{Q}[y,z]$, the ring R is coherent but $R[x]$ is not coherent according to a result of Soublin [25]. Given a finitely generated ideal I of a coherent ring R, the quotient ring R/I is coherent.

If R is a coherent ring such that the polynomial ring $R[x_1, \ldots, x_n]$ is coherent, then the projective n-space $\mathbb{P}^n_R = \mathsf{Proj}\, R[x_0, \ldots, x_n]$ over R is a coherent scheme. Indeed, $\mathbb{P}^n_R$ is quasi-compact and quasi-separated by [9, 5.1] and covered by $\mathsf{Spec}\, R[x_0/x_i, \ldots, x_n/x_i]$ with each $R[x_0/x_i, \ldots, x_n/x_i]$ coherent by assumption.

Below we shall need the following result.

Theorem 9.1 (Herzog [13], Krause [19]). *Let $\mathcal{C}$ be a locally coherent Grothendieck category. There is a bijective correspondence between the Serre subcategories $\mathcal{P}$ of* $\operatorname{coh}\mathcal{C}$ *and the localizing subcategories $\mathcal{S}$ of $\mathcal{C}$ of finite type. This correspondence is given by the functions*

$$\mathcal{P} \longmapsto \vec{\mathcal{P}} = \{\varinjlim C_i \mid C_i \in \mathcal{P}\}$$
$$\mathcal{S} \longmapsto \mathcal{S} \cap \operatorname{coh}\mathcal{C}$$

which are mutual inverses.

Proposition 9.2. *Let X be a quasi-compact and quasi-separated scheme. Then X is a coherent scheme if and only if* $\operatorname{coh}(X)$ *is an abelian category or, equivalently,* $\operatorname{Qcoh}(X)$ *is a locally coherent Grothendieck category.*

Proof. Suppose X is a coherent scheme. We have to show that every finitely generated subobject $\mathcal{F}$ of a finitely presented object $\mathcal{G}$ is finitely presented. It follows from [11, I.6.9.10] and Proposition 3.1 that $\mathcal{F} \in \operatorname{fg}(\operatorname{Qcoh}(X))$ if and only if it is locally finitely generated.

Given $P \in X$ there is an open subset U of P and an exact sequence

$$\mathcal{O}_U^n \to \mathcal{O}_U^m \to \mathcal{G}|_U \to 0.$$

By assumption, there is an affine neighbourhood $\operatorname{Spec} R$ of P with R a coherent ring. Let $f \in R$ be such that $P \in D(f) \subseteq \operatorname{Spec} R \cap U$, where $D(f) = \{Q \in \operatorname{Spec} R \mid f \notin Q\}$. One has $\mathcal{O}_X(D(f)) = \mathcal{O}_R(D(f)) = R_f$, hence we get an exact sequence

$$\mathcal{O}_{R_f}^n \to \mathcal{O}_{R_f}^m \to \mathcal{G}|_{D(f)} \to 0.$$

Since R is a coherent ring then so is R_f.

There is an open neighbourhood V of P and an epimorphism $\mathcal{O}_V^k \twoheadrightarrow \mathcal{F}|_V$, $k \in \mathbb{N}$. Without loss of generality, we may assume that $V = D(f)$ for some $f \in R$. It follows that $\mathcal{F}|_{D(f)} \subset \mathcal{G}|_{D(f)}$ is a finitely presented $\mathcal{O}_{R_f}$-module, because R_f is a coherent ring. Therefore $\mathcal{F}$ is locally finitely presented, and hence $\mathcal{F} \in \operatorname{fp}(\operatorname{Qcoh}(X))$.

Now suppose that $\operatorname{Qcoh}(X)$ is a locally coherent Grothendieck category. Given $P \in X$ and an affine neighbourhood $\operatorname{Spec} R$ of P, we want to show that R is a coherent ring. The localizing subcategory $\mathcal{S} = \{\mathcal{F} \mid \mathcal{F}|_{\operatorname{Spec} R} = 0\}$ is of finite type, and therefore $\operatorname{Qcoh}(X)/\mathcal{S}$ is a locally coherent Grothendieck category. It follows from Proposition 2.1 that $\operatorname{Mod} R \cong \operatorname{Qcoh}(\operatorname{Spec} R) \cong \operatorname{Qcoh}(X)/\mathcal{S}$ is a locally coherent Grothendieck category, whence R is coherent. □

Theorem 9.3 (Classification). *Let X be a coherent scheme. Then the maps*

$$V \mapsto \mathcal{S} = \{\mathcal{F} \in \operatorname{coh}(X) \mid \operatorname{supp}_X(\mathcal{F}) \subseteq V\}$$

and

$$\mathcal{S} \mapsto V = \bigcup_{\mathcal{F} \in \mathcal{S}} \operatorname{supp}_X(\mathcal{F})$$

induce bijections between

(1) *the set of all subsets of the form $V = \bigcup_{i \in \Omega} V_i$ with quasi-compact open complement $X \setminus V_i$ for all $i \in \Omega$,*
(2) *the set of all tensor Serre subcategories in* $\operatorname{coh}(X)$.

Theorem 9.4. *Let X be a coherent scheme. The assignments*

$$\mathcal{T} \mapsto \mathcal{S} = \{\mathcal{F} \in \operatorname{coh}(X) \mid \operatorname{supp}_X(\mathcal{F}) \subseteq \bigcup_{n \in \mathbb{Z}, E \in \mathcal{T}} \operatorname{supp}_X(H_n(E))\}$$

and

$$\mathcal{S} \mapsto \{E \in \mathcal{D}_{\mathrm{per}}(X) \mid H_n(E) \in \mathcal{S} \text{ for all } n \in \mathbb{Z}\}$$

induce a bijection between

(1) *the set of all tensor thick subcategories of* $\mathcal{D}_{\mathrm{per}}(X)$,
(2) *the set of all tensor Serre subcategories in* $\mathrm{coh}(X)$.

Let X be a coherent scheme. The ringed space $(\mathsf{Spec}(\mathrm{coh}(X)), \mathcal{O}_{\mathrm{coh}(X)})$ is introduced similar to $(\mathsf{Spec}(\mathrm{Qcoh}(X)), \mathcal{O}_{\mathrm{Qcoh}(X)})$.

Theorem (Reconstruction)**.** *Let X be a coherent scheme. Then there is a natural isomorphism of ringed spaces*

$$f : (X, \mathcal{O}_X) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{coh}(X)), \mathcal{O}_{\mathrm{coh}(X)}).$$

The theorems are direct consequences of the corresponding theorems for quasi-compact, quasi-separated schemes and Theorem 9.1. The interested reader can check these without difficulty.

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